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LECTURES ON LYAPUNOV EXPONENTS

The theory of Lyapunov exponents originated over a century ago in the study of the stability of solutions of differential equations. Written by one of the subject's leading authorities, this book is both an account of the classical theory, from a modern view, and an introduction to the significant developments relating the subject to dynamical systems, ergodic theory, mathematical physics and probability. It is based on the author's own graduate course and is reasonably self-contained with an extensive set of exercises provided at the end of each chapter.

This book makes a welcome addition to the literature, serving as a graduate text and a valuable reference for researchers in the field.

Marcelo Viana is Professor of Mathematics at the Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro.

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Lectures on Lyapunov Exponents

MARCELO VIANA
Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro

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To Tania, Miguel and Anita,
for their understanding.

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Preface

1. The study of characteristic exponents originated from the fundamental work of Aleksandr Mikhailovich Lyapunov [85] on the stability of solutions of differential equations. Consider a linear equation

$$\dot{v}(t) = B(t) \cdot v(t) \tag{1}$$

where $B(\cdot)$ is a bounded function from $\mathbb{R}$ to the space of $d \times d$ matrices. By the general theory of differential equations, there exists a so-called *fundamental matrix* A^t, $t \in \mathbb{R}$ such that $v(t) = A^t \cdot v_0$ is the unique solution of (1) with initial condition $v(0) = v_0$. If the *characteristic exponents*

$$\lambda(v) = \limsup_{t\to\infty} \frac{1}{t} \log \|A^t \cdot v\| \tag{2}$$

are negative, for all $v \neq 0$, then the trivial solution $v(t) \equiv 0$ is asymptotically stable, and even exponentially asymptotically stable. The stability theorem of Lyapunov asserts that, under an additional regularity condition, stability remains valid for nonlinear perturbations

$$\dot{w}(t) = B(t) \cdot w(t) + F(t,w) \quad \text{with } \|F(t,w)\| \leq \text{const}\, \|w\|^{1+\varepsilon}.$$

That is, the trivial solution $w(t) \equiv 0$ is still exponentially asymptotically stable.

The regularity condition of Lyapunov means, essentially, that the limit in (2) does exist, even if one replaces vectors v by l-vectors $v_1 \wedge \cdots \wedge v_l$; that is, elements of the k-exterior power of $\mathbb{R}^d$, for any $0 \leq l \leq d$. This is usually difficult to check in specific situations. But the multiplicative ergodic theorem of Oseledets asserts that Lyapunov regularity holds with full probability, in great generality. In particular, it holds on almost every flow trajectory, relative to any probability measure invariant under the flow.

2. The work of Furstenberg, Kesten, Oseledets, Kingman, Ledrappier, Guivarc'h, Raugi, Gol'dsheid, Margulis and other mathematicians, mostly in the

1960s–80s, built the study of Lyapunov characteristic exponents into a very active research field in its own right, and one with an unusually vast array of interactions with other areas of Mathematics and Physics, such as stochastic processes (random matrices and, more generally, random walks on groups), spectral theory (Schrödinger-type operators) and smooth dynamics (non-uniform hyperbolicity), to mention just a few.

My own involvement with the subject goes back to the late 20th century and was initially motivated by my work with Christian Bonatti and José F. Alves on the ergodic theory of partially hyperbolic diffeomorphisms and, soon afterwards, with Jairo Bochi on the dependence of Lyapunov exponents on the underlying dynamical system. The way these two projects unfolded very much inspired the choice of topics in the present book.

3. A diffeomorphism $f : M \to M$ is called *partially hyperbolic* if there exists a Df-invariant decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

of the tangent bundle such that E^s is uniformly contracted and E^u is uniformly expanded by the derivative Df, whereas the behavior of Df along the *center bundle* E^c lies somewhere in between. It soon became apparent that to improve our understanding of such systems one should try to get a better hold of the behavior of $Df \mid E^c$ and, in particular, of its Lyapunov exponents. In doing this, we turned to the classical linear theory for inspiration.

That program proved to be very fruitful, as much in the linear context (e.g. the proof of the Zorich–Kontsevich conjecture, by Artur Avila and myself) as in the setting of partially hyperbolic dynamics we had in mind originally (e.g the rigidity results by Artur Avila, Amie Wilkinson and myself), and remains very active to date, with important contributions from several mathematicians.

4. Before that, in the early 1980s, Ricardo Mañé came to the surprising conclusion that generic (a residual subset of) volume-preserving C^1 diffeomorphisms on any surface have zero Lyapunov exponents, or else they are globally hyperbolic (Anosov); in fact, the second alternative is possible only if the surface is the torus $\mathbb{T}^2$. This discovery went against the intuition drawn from the classical theory of Furstenberg.

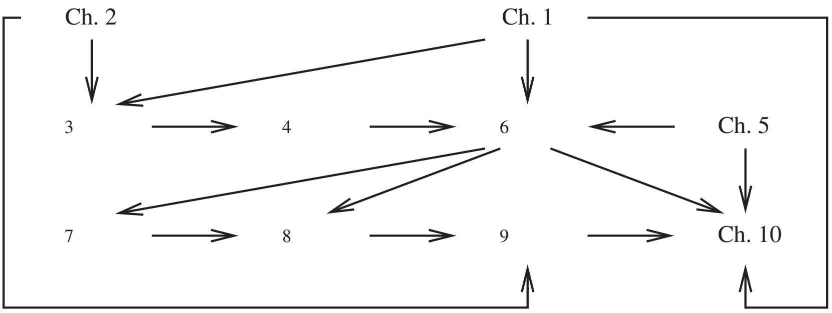
Although Mañé did not write a complete proof of his findings, his approach was successfully completed by Bochi almost two decades later. Moreover, the conclusions were extended to arbitrary dimension, both in the volume-preserving and in the symplectic case, by Bochi and myself.

5. In this monograph I have sought to cover the fundamental aspects of the classical theory (mostly in Chapters 1 through 6), as well as to introduce some of the more recent developments (Chapters 7 through 10).

The text started from a graduate course that I taught at IMPA during the (southern hemisphere) summer term of 2010. The very first draft consisted of lecture notes taken by Carlos Bocker, José Régis Varão and Samuel Feitosa. The unpublished notes [9] and [28], by Artur Avila and Jairo Bochi were important for setting up the first part of the course.

The material was reviewed and expanded later that year, in my seminar, with the help of graduate students and post-docs of IMPA's Dynamics group. I taught the course again in early 2014, and I took that occasion to add some proofs, to reorganize the exercises and to include historic notes in each of the chapters. Chapter 10 was completely rewritten and this preface was also much expanded.

6. The diagram below describes the logical connections between the ten chapters. The first two form an introductory cycle. In Chapter 1 we offer a glimpse of what is going to come by stating three main results, whose proofs will appear, respectively, in Chapters 3, 6 and 10. In Chapter 2 we introduce the notion of linear cocycle, upon which is built the rest of the text. We examine more closely the particular case of hyperbolic cocycles, especially in dimension 2, as this will be useful in Chapter 9.



In the next four chapters we present the main classical results, including the Furstenberg–Kesten theorem and the subadditive ergodic theorem of Kingman (Chapter 3), the multiplicative ergodic theorem of Oseledets (Chapter 4), Ledrappier's exponent representation theorem, Furstenberg's formula for exponents of irreducible cocycles and Furstenberg's simplicity theorem in dimension 2 (Chapter 6). The proof of the multiplicative ergodic theorem is based on the subadditive ergodic theorem and also heralds the connection between Lyapunov exponents and invariant/stationary measures that lies at the heart of the results in Chapter 6. In Chapter 5 we provide general tools to develop that connection, in both the invertible and the non-invertible case.

7. The last four chapters are devoted to more advanced material. The main goal there is to provide a friendly introduction to the existing research literature. Thus, the emphasis is on transparency rather than generality or completeness. This means that, as a rule, we choose to state the results in the simplest possible (yet relevant) setting, with suitable references given for stronger statements.

Chapter 7 introduces the invariance principle and exploits some of its consequences, in the context of locally constant linear cocycles. This includes Furstenberg's criterion for $\lambda_- = \lambda_+$, that extends Furstenberg's simplicity theorem to arbitrary dimension. The invariance principle has been used recently to analyze much more general dynamical systems, linear and nonlinear, whose Lyapunov exponents vanish. A finer extension of Furstenberg's theorem appears in Chapter 8, where we present a criterion for simplicity of the whole Lyapunov spectrum.

Then, in Chapter 9, we turn our attention to the contrasting Mañé–Bochi phenomenon of systems whose Lyapunov spectra are generically *not* simple. We prove an instance of the Mañé–Bochi theorem, for continuous linear cocycles. Moreover, we explain how those methods can be adapted to construct examples of discontinuous dependence of Lyapunov exponents on the cocycle, even in the Hölder-continuous category. Having raised the issue of (dis)continuity, in Chapter 10 we prove that for products of random matrices in $\mathrm{GL}(2)$ the Lyapunov exponents do depend continuously on the cocycle data.

8. Each chapter ends with set of notes and a list of exercises. Some of the exercises are actually used in the proofs. They should be viewed as an invitation for the reader to take an active part in the arguments. Throughout, it is assumed that the reader is familiar with the basic ideas of Measure Theory, Differential Topology and Ergodic Theory. All that is needed can be found, for instance, in my book with Krerley Oliveira, *Fundamentos da Teoria Ergódica* [114]; a translation into English is under way.

I thank David Tranah, of Cambridge University Press, for his interest in this book and for patiently waiting for the writing to be completed. I am also grateful to Vaughn Climenhaga, and David himself, for a careful revision of the manuscript that very much helped improve the presentation.

Rio de Janeiro, March, 2014
Marcelo Viana