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Universality conjecture for all Airy, sine and Bessel kernels in the complex plane

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We address the question of how the celebrated universality of local correlations for the real eigenvalues of Hermitian random matrices of size $N \times N$ can be extended to complex eigenvalues in the case of random matrices without symmetry. Depending on the location in the spectrum, particular large-N limits (the so-called weakly non-Hermitian limits) lead to one-parameter deformations of the Airy, sine and Bessel kernels into the complex plane. This makes their universality highly suggestive for all symmetry classes. We compare all the known limiting real kernels and their deformations into the complex plane for all three Dyson indices $\beta = 1, 2, 4$, corresponding to real, complex and quaternion real matrix elements. This includes new results for Airy kernels in the complex plane for $\beta = 1, 4$. For the Gaussian ensembles of elliptic Ginibre and non-Hermitian Wishart matrices we give all kernels for finite N, built from orthogonal and skew-orthogonal polynomials in the complex plane. Finally we comment on how much is known to date regarding the universality of these kernels in the complex plane, and discuss some open problems.

1. Introduction

The topic of universality in Hermitian random matrix theory (RMT) has attracted a lot of attention in the mathematics community recently, particularly in the context of matrices with elements that are independent random variables, as reviewed in [Tao and Vu 2012; Erdős and Yau 2012]. The question that one tries to answer is this: under what conditions are the statistics of eigenvalues of $N \times N$ matrices with independent Gaussian variables the same (for large matrices) as for more general RMT where matrix elements may become coupled? This has been answered under very general assumptions, and we refer to some recent reviews on invariant [Kuijlaars 2011; Deift and Gioev 2009] and noninvariant [Tao and Vu 2012; Erdős and Yau 2012] ensembles.

In this short note we would like to advocate the idea that non-Hermitian RMT with eigenvalues in the complex plane also warrants the investigation of universality. Apart from the interest in its own right, these models have important applications in physics and other sciences (see, e.g., [Akemann et al. 2011a]). We will focus here on RMT that is close to Hermitian, a regime that is particularly important for applications in quantum chaotic scattering (see [Fyodorov and Sommers 2003] for a review) and quantum chromodynamics (QCD), for example. In the latter case, the non-Hermiticity may arise from describing the effect of quark chemical potential (as reviewed in [Verbaarschot 2011; Akemann 2007]), or from finite lattice spacing effects of the Wilson–Dirac operator (see [Damgaard et al. 2010; Akemann et al. 2011b] as well as [Kieburg 2012] for the solution of this non-Hermitian RMT).

Being a system of N coupled eigenvalues, Hermitian RMT already offers a rich variety of large-N limits, where one has to distinguish the bulk and (soft) edge of the spectrum for Wigner–Dyson (WD) ensembles, and in addition the origin (hard edge) for Wishart–Laguerre (WL, or chiral) RMT. Not surprisingly complex eigenvalues offer even more possibilities. The limit we will investigate is known as the weakly non-Hermitian regime; it *connects* Hermitian and (strongly) non-Hermitian RMT, and was first introduced in [Fyodorov et al. 1997a; 1997b] in the bulk of the spectrum. For strong non-Hermiticity — which includes the well-known circular law and the corresponding universality results — we refer to [Khoruzhenko and Sommers 2011] and references therein, although the picture there is also far from being complete; an important breakthrough was published recently in [Tao and Vu 2014].

In the next section we give a brief list of the six non-Hermitian WD and WL ensembles, and indicate where they were first solved in the weak limit. There are three principal reasons why we believe that universality may hold. First, in some cases two different Gaussian RMT both give the same answers. Second, there are heuristic arguments available for i.i.d. matrix elements using supersymmetry [Fyodorov et al. 1998], as well as for invariant non-Gaussian ensembles using large-N factorisation and orthogonal polynomials (OP) [Akemann 2002]. Third, the resulting limiting kernels of (skew-) OP look very similar to the corresponding kernels of real eigenvalues, being merely one-parameter deformations of them. One of the main goals of this paper is to illustrate this fact. For this purpose we give a complete list of all the known Airy, sine, and Bessel kernels for real eigenvalues, side-by-side with their deformed kernels in the complex plane, where some of our results are new.

2. Random matrices and their limiting kernels

In this section we briefly introduce the Gaussian random matrix ensembles that we consider, and give a list of the limiting kernels they lead to, for both real and complex eigenvalues. For simplicity we have restricted ourselves to Gaussian ensembles in the Hermitian cases, in order to highlight the parallels to their non-Hermitian counterparts.

We begin with the classical WD and Ginibre ensembles in Section 2.1, displaying Airy (Section 2.2) and sine (Section 2.3) behaviour at the (soft) edge and in the bulk of the spectrum respectively, as well as their deformations. We then introduce the WL ensembles and their non-Hermitian counterparts in Section 2.4, in order to access the Bessel behaviour (Section 2.5) at the origin (or hard edge). The corresponding orthogonal and skew-orthogonal Hermite and Laguerre polynomials are given in Appendix A, and precise statements of the limits that lead to the microscopic kernels can be found in Appendix B.

2.1. ***Gaussian ensembles with eigenvalues on*** $\mathbb{R}$ ***and*** $\mathbb{C}$***.*** The three classical Gaussian Wigner–Dyson ensembles (the GOE, GUE and GSE) are defined as [Mehta 2004]

$$\begin{aligned}\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}} &= \int dH \exp[-\beta \operatorname{Tr} H^2/4] \\ &= c_{N,\beta} \prod_{j=1}^{N} \int_{\mathbb{R}} dx_j\, w_\beta(x_j)\, |\Delta_N(\{x\})|^\beta. \end{aligned} \tag{2.1}$$

The random matrix elements H_{kl} are real, complex, or quaternion real numbers for $\beta = 1, 2, 4$ respectively, with the condition that the $N \times N$ matrix H (N is taken to be even for simplicity) is real symmetric, complex Hermitian or complex Hermitian and self-dual for $\beta = 1, 2, 4$. In the first equation we integrate over all independent matrix elements denoted by dH. The Gaussian weight completely factorises and thus the independent elements are normal random variables; for $\beta = 1$, for example, the real elements are distributed $\mathcal{N}(0, 1)$ for off-diagonal elements, and $\mathcal{N}(0, 2)$ for diagonal elements.

In the second equality of (2.1), we diagonalised the matrix

$$H = U \operatorname{diag}(x_1, \ldots, x_N)\, U^{-1},$$

where U is an orthogonal, unitary or unitary-symplectic matrix for $\beta = 1, 2, 4$. The integral over U factorises and leads to the known constants $c_{N,\beta}$. We obtain a Gaussian weight $w_\beta(x)$ and the Vandermonde determinant $\Delta_N(\{x\})$ from the Jacobian of the diagonalisation,

$$w_\beta(x) = \exp[-\beta x^2/4], \qquad \Delta_N(\{x\}) = \prod_{1\le l<k\le N} (x_k - x_l). \tag{2.2}$$

The integrand on the right-hand side of (2.1) times $c_{N,\beta}/\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}}$ defines the normalised joint probability distribution function (jpdf) of all eigenvalues. The k-point correlation function R_k^β, which is proportional to the jpdf integrated over

$N-k$ eigenvalues, can be expressed through a single kernel $K_N^{\beta=2}$ of orthogonal polynomials (OP) for $\beta=2$, or through a 2×2 matrix-valued kernel involving skew-OP for $\beta=1,4$:

$$R_k^{\beta=2}(x_1,\ldots,x_k)=\det_{i,j=1,\ldots,k}[K_N^{\beta=2}(x_i,x_j)],$$
$$R_k^{\beta=1,4}(x_1,\ldots,x_k)=\mathop{\mathrm{Pf}}_{i,j=1,\ldots,k}\left[\begin{pmatrix}K_N^{\beta=1,4}(x_i,x_j) & -G_N^{\beta=1,4}(x_i,x_j)\\ G_N^{\beta=1,4}(x_j,x_i) & -W_N^{\beta=1,4}(x_i,x_j)\end{pmatrix}\right]. \tag{2.3}$$

The matrix kernel elements K_N and W_N are not independent of G_N but are related by differentiation and integration, respectively. These relations will be given later for the limiting kernels.

The three parameter-dependent Ginibre (i.e., elliptic or Ginibre–Girko) ensembles, denoted by GinOE, GinUE, and GinSE, can be written as

$$\mathcal{Z}_N^{\mathrm{Gin}\beta\mathrm{E}}(\tau)=\int dJ\exp\left[\frac{-\gamma_\beta}{1-\tau^2}\mathrm{Tr}\left(JJ^\dagger-\frac{\tau}{2}(J^2+J^{\dagger\,2})\right)\right]$$
$$=\int dH_1\,dH_2\exp\left[-\frac{\gamma_\beta\,\mathrm{Tr}\,H_1^2}{1+\tau}-\frac{\gamma_\beta\,\mathrm{Tr}\,H_2^2}{1-\tau}\right], \tag{2.4}$$

with $\tau\in[0,1)$. We use the parametrisation of [Khoruzhenko and Sommers 2011], with $\gamma_{\beta=2}=1$ and $\gamma_{\beta=1,4}=\frac{1}{2}$. The matrix elements of J are of the same types as for H for all three values of β, but without any further symmetry constraint. Decomposing $J=H_1+iH_2$ into its Hermitian and anti-Hermitian parts, these ensembles can be viewed as Gaussian two-matrix models. For $\tau=0$ (maximal non-Hermiticity) the distribution for all matrix elements again factorises. In the opposite, that is, Hermitian, limit ($\tau\to 1$), the parameter-dependent Ginibre ensembles become the Wigner–Dyson ensembles. The jpdf of complex (and real) eigenvalues can be computed by transforming J into the following form, $J=U(Z+T)U^{-1}$. For $\beta=2$ this is the Schur decomposition, with $Z=\mathrm{diag}(z_1,\ldots,z_N)$ containing the complex eigenvalues, and T being upper triangular:[1]

$$\mathcal{Z}_N^{\mathrm{GinUE}}(\tau)=c_{N,\mathbb{C}}^{\beta=2}\prod_{j=1}^{N}\int_{\mathbb{C}}d^2z_j\,w_{\beta=2}^{\mathbb{C}}(z_j)\,|\Delta_N(\{z\})|^2,$$
$$w_{\beta=2}^{\mathbb{C}}(z)=\exp\left[\frac{-1}{1-\tau^2}\left(|z|^2-\frac{\tau}{2}(z^2+z^{*\,2})\right)\right]. \tag{2.5}$$

For $\beta=1,4$ we follow [Khoruzhenko and Sommers 2011] where the two ensembles have been cast into a unifying framework. For simplicity we choose

[1] The resulting jpdf of complex eigenvalues for normal matrices with $T\equiv 0$ at $\beta=2$ is the same.

N to be even. Here the matrix Z can be chosen to be 2×2 block diagonal and T to be upper block triangular. The calculation of the jpdf reduces to a 2×2 calculation, yielding

$$\mathcal{Z}_N^{\mathrm{GinO/SE}}(\tau) = c_{N,\mathbb{C}}^{\beta=1,4} \prod_{j=1}^{N} \int_{\mathbb{C}} d^2 z_j \prod_{k=1}^{N/2} \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_{2k-1}, z_{2k})\, \Delta_N(\{z\}), \tag{2.6}$$

where we have introduced an antisymmetric bivariate weight function. For $\beta = 1$, this is given by

$$\begin{aligned}\mathcal{F}_{\beta=1}^{\mathbb{C}}(z_1, z_2) &= w_{\beta=1}^{\mathbb{C}}(z_1) w_{\beta=1}^{\mathbb{C}}(z_2)\\ &\quad\times \left(2i\delta^2(z_1 - z_2^*)\operatorname{sign}(y_1) + \delta^1(y_1)\delta^1(y_2)\operatorname{sign}(x_2 - x_1)\right),\\ (w_{\beta=1}^{\mathbb{C}}(z))^2 &= \operatorname{erfc}\left(\frac{|z - z^*|}{\sqrt{2(1-\tau^2)}}\right)\exp\left[\frac{-1}{2(1+\tau)}(z^2 + z^{*\,2})\right],\end{aligned} \tag{2.7}$$

and for $\beta = 4$ by

$$\begin{aligned}\mathcal{F}_{\beta=4}^{\mathbb{C}}(z_1, z_2) &= w_{\beta=4}^{\mathbb{C}}(z_1) w_{\beta=4}^{\mathbb{C}}(z_2)\,(z_1 - z_2)\,\delta(z_1 - z_2^*),\\ (w_{\beta=4}^{\mathbb{C}}(z))^2 &= w_{\beta=2}^{\mathbb{C}}(z).\end{aligned} \tag{2.8}$$

For $\beta = 1$, it should be noted that the integrand in (2.6) is not always positive, and so a symmetrisation must be applied when determining the correlation functions below.[2] For $\beta = 4$, the parameter N in (2.6) should — in our convention — be taken to be the size of the complex-valued matrix that is equivalent to the original quaternion real matrix.

The correlation functions can be written in a similar form as for the real eigenvalues

$$\begin{aligned}R_{k,\mathbb{C}}^{\beta=2}(z_1, \dots, z_k) &= \det_{i,j=1,\dots,k}[K_{N,\mathbb{C}}^{\beta=2}(z_i, z_j^*)],\\ R_{k,\mathbb{C}}^{\beta=1,4}(z_1, \dots, z_k) &= \operatorname*{Pf}_{i,j=1,\dots,k}\left[\begin{pmatrix} K_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) & -G_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j)\\ G_{N,\mathbb{C}}^{\beta=1,4}(z_j, z_i) & -W_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j)\end{pmatrix}\right],\end{aligned} \tag{2.9}$$

where the elements of the matrix kernels are related through

$$\begin{aligned}G_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) &= -\int_{\mathbb{C}} d^2 z\, K_{N,\mathbb{C}}^{\beta=1,4}(z_i, z)\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z, z_j),\\ W_{N,\mathbb{C}}^{\beta=1,4}(z_i, z_j) &= \int_{\mathbb{C}^2} d^2 z\, d^2 z'\, \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i, z) K_{N,\mathbb{C}}^{\beta=1,4}(z, z')\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z', z_j)\\ &\quad - \mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i, z_j).\end{aligned} \tag{2.10}$$

[2] It is, however, possible to write the partition function $\mathcal{Z}_N^{\mathrm{GinOE}}$ as an integral over a true (i.e., positive) jpdf, by, for example, appropriately ordering the eigenvalues; however, such a representation is technically more difficult to work with.

The kernels $K^{\beta}_{N,\mathbb{C}}(z,z')$ are given explicitly in Appendix A.

For $\beta=1$, we can write

$$G^{\beta=1}_{N,\mathbb{C}}(z_1,z_2)=\delta^1(y_2)G^{\beta=1}_{N,\mathbb{C},\mathrm{real}}(z_1,x_2)+G^{\beta=1}_{N,\mathbb{C},\mathrm{com}}(z_1,z_2),$$

$$\begin{aligned}W^{\beta=1}_{N,\mathbb{C}}(z_1,z_2)=\delta^1(y_1)\delta^1(y_2)W^{\beta=1}_{N,\mathbb{C},\mathrm{real,real}}(x_1,x_2)\\ +\delta^1(y_1)W^{\beta=1}_{N,\mathbb{C},\mathrm{real,com}}(x_1,z_2)+\delta^1(y_2)W^{\beta=1}_{N,\mathbb{C},\mathrm{com,real}}(z_1,x_2)\\ +W^{\beta=1}_{N,\mathbb{C},\mathrm{com,com}}(z_1,z_2)-\mathcal{F}^{\mathbb{C}}_{\beta=1}(z_1,z_2),\end{aligned} \tag{2.11}$$

whereas, for $\beta=4$, (2.8) implies the following relations:

$$\begin{aligned}G^{\beta=4}_{N,\mathbb{C}}(z_1,z_2)&=(z_2-z_2^*)w^{\mathbb{C}}_{\beta=2}(z_2)K^{\beta=4}_{N,\mathbb{C}}(z_1,z_2^*),\\ W^{\beta=4}_{N,\mathbb{C}}(z_1,z_2)&=-(z_1-z_1^*)(z_2-z_2^*)w^{\mathbb{C}}_{\beta=2}(z_1)w^{\mathbb{C}}_{\beta=2}(z_2)K^{\beta=4}_{N,\mathbb{C}}(z_1^*,z_2^*),\end{aligned} \tag{2.12}$$

where in the final expression we have dropped the term representing the perfect correlation between an eigenvalue z and its complex conjugate z^*. For this reason, for $\beta=4$ we will only give one of the matrix kernel elements in the following.

Note that $\beta=1$ is special as the eigenvalues of a real asymmetric matrix are either real or come in complex conjugate pairs. Therefore we will have to distinguish kernels (and k-point densities) of real, complex or mixed arguments.

In order to specify the limiting kernels we first need the behaviour of the mean (or macroscopic) spectral density. At large N, and for all three values of β, the (real) eigenvalues in the Hermitian ensembles are predominantly concentrated within the Wigner semicircle $\rho_{\mathrm{sc}}(x)=(2\pi N)^{-1}\sqrt{4N-x^2}$ on $[-2\sqrt{N},2\sqrt{N}]$, whereas in the non-Hermitian ensemble, the complex eigenvalues lie mostly within an ellipse with half-axes of lengths $(1+\tau)\sqrt{N}$ and $(1-\tau)\sqrt{N}$, with constant density $\rho_{\mathrm{el}}(z)=(N\pi(1-\tau^2))^{-1}$. Depending on where (and how) we magnify the spectrum locally, we obtain different asymptotic Airy or sine kernels for each $\beta=1,2,4$. In the following we will give all of the known real kernels; see [Kuijlaars 2011], for example, for a complete list and references, together with their deformations into the complex plane. For the Bessel kernels which will be introduced later we need to consider different matrix ensembles, see Section 2.4 below.

2.2. ***Limiting Airy kernels on $\mathbb{R}$ and $\mathbb{C}$.*** When appropriately zooming into the "square root" edge of the semicircle, the three well-known Airy kernels (matrix-valued for $\beta=1,4$) are obtained for real eigenvalues. For complex eigenvalues we have to consider the vicinity of the eigenvalues on a thin ellipse which have the largest real parts, and where the weakly non-Hermitian limit introduced in [Bender 2010] is defined such that

$$\sigma=N^{\frac{1}{6}}\sqrt{1-\tau} \tag{2.13}$$

remains fixed (see Appendix B for the precise details of the scaling of the eigenvalues). This leads to one-parameter deformations of the Airy kernels in the complex plane. Whilst the results for $\beta = 2$ are already known [Bender 2010; Akemann and Bender 2010], our results for $\beta = 1, 4$, stated below, are new [Akemann and Phillips 2014]:

$\underline{\beta = 2}$:

$$K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2) = \frac{\mathrm{Ai}(x_1)\,\mathrm{Ai}'(x_2) - \mathrm{Ai}'(x_1)\,\mathrm{Ai}(x_2)}{x_1 - x_2} = \int_0^\infty dt\,\mathrm{Ai}(x_1 + t)\,\mathrm{Ai}(x_2 + t), \tag{2.14}$$

$$K_{\mathrm{Ai},\mathbb{C}}^{\beta=2}(z_1, z_2) = \frac{1}{\sigma\sqrt{\pi}}\exp\left(-\frac{y_1^2 + y_2^2}{2\sigma^2} + \frac{\sigma^6}{6} + \frac{\sigma^2(z_1 + z_2)}{2}\right) \times \int_0^\infty dt\, e^{\sigma^2 t}\,\mathrm{Ai}\Big(z_1 + t + \frac{\sigma^4}{4}\Big)\,\mathrm{Ai}\Big(z_2 + t + \frac{\sigma^4}{4}\Big). \tag{2.15}$$

In the Hermitian limit $\sigma \to 0$ we obtain

$$K_{\mathrm{Ai},\mathbb{C}}^{\beta=2}(z_1, z_2) \to \sqrt{\delta^1(y_1)\delta^1(y_2)}\,K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2),$$

with the factor in front of the integral in (2.15) projecting the imaginary parts of the eigenvalues to zero. For the integral itself — which is obtained from the limit of the sum of the OP on $\mathbb{C}$ given in (A.4) — the deformation in σ is very smooth. The same deformed Airy kernel can be obtained from the corresponding WL ensemble equation (2.29) [Akemann and Bender 2010] with kernel equation (A.5), and is thus universal.

$\underline{\beta = 4}$:

$$\begin{aligned} G_{\mathrm{Ai}}^{\beta=4}(x_1,x_2) &= -\tfrac{1}{2}K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2) + \tfrac{1}{4}\,\mathrm{Ai}(x_1)\int_{x_2}^\infty dt\,\mathrm{Ai}(t),\\ K_{\mathrm{Ai}}^{\beta=4}(x_1, x_2) &= \frac{\partial}{\partial x_2}G_{\mathrm{Ai}}^{\beta=4}(x_1, x_2),\\ W_{\mathrm{Ai}}^{\beta=4}(x_1, x_2) &= -\int_{x_1}^\infty ds\, G_{\mathrm{Ai}}^{\beta=4}(s, x_2) \qquad (2.16)\\ &= -\tfrac{1}{4}\int_0^\infty ds\int_0^s dt\Big(\mathrm{Ai}(x_2{+}t)\,\mathrm{Ai}(x_1{+}s) - \mathrm{Ai}(x_2{+}s)\,\mathrm{Ai}(x_1{+}t)\Big), \end{aligned}$$

$$G_{\mathrm{Ai},\mathbb{C}}^{\beta=4}(z_1,z_2)=\frac{iy_2}{4\sigma^3\sqrt{\pi}}\exp\left(-\frac{y_1^2+y_2^2}{2\sigma^2}+\frac{\sigma^6}{6}+\frac{\sigma^2(z_1+z_2^*)}{2}\right)\\ \times\int_0^\infty ds\int_0^s dt\, e^{\frac{1}{2}\sigma^2(s+t)}\\ \times\left(\mathrm{Ai}\left(z_2^*+s+\frac{\sigma^4}{4}\right)\mathrm{Ai}\left(z_1+t+\frac{\sigma^4}{4}\right)-(z_1\leftrightarrow z_2^*)\right). \quad (2.17)$$

The integral in (2.17) which is also present in the other two kernel elements — see (2.12) — clearly reduces to that in (2.16) in the Hermitian limit, whereas the prefactors provide the appropriate Dirac delta functions. When analysing the Hermitian limit in detail, the real kernel elements $G_{\mathrm{Ai}}^{\beta=4}$ and $K_{\mathrm{Ai}}^{\beta=4}$ follow from a Taylor expansion of $W_{\mathrm{Ai},\mathbb{C}}^{\beta=4}$; see [Akemann and Basile 2007] for a discussion of the analogous Hermitian limit of the Bessel kernel.

$\underline{\beta=1}$:

$$G_{\mathrm{Ai}}^{\beta=1}(x_1,x_2)=-\int_0^\infty dt\,\mathrm{Ai}(x_1+t)\,\mathrm{Ai}(x_2+t)-\tfrac{1}{2}\mathrm{Ai}(x_1)\left(1-\int_{x_2}^\infty dt\,\mathrm{Ai}(t)\right),$$

$$K_{\mathrm{Ai}}^{\beta=1}(x_1,x_2)=\frac{\partial}{\partial x_2}G_{\mathrm{Ai}}^{\beta=1}(x_1,x_2),$$

$$W_{\mathrm{Ai}}^{\beta=1}(x_1,x_2)=-\int_{x_1}^\infty ds\,G_{\mathrm{Ai}}^{\beta=1}(s,x_2)-\frac{1}{2}\int_{x_1}^{x_2}dt\,\mathrm{Ai}(t)\\ +\frac{1}{2}\int_{x_1}^\infty ds\,\mathrm{Ai}(s)\int_{x_2}^\infty dt\,\mathrm{Ai}(t)-\tfrac{1}{2}\,\mathrm{sign}(x_1-x_2), \quad (2.18)$$

$$G_{\mathrm{Ai},\mathbb{C},\mathrm{real}}^{\beta=1}(x_1,x_2)=-\exp\left(\frac{\sigma^6}{6}+\frac{\sigma^2(x_1+x_2)}{2}\right)\\ \times\int_0^\infty dt\,\mathrm{e}^{\sigma^2 t}\,\mathrm{Ai}\left(x_1+t+\frac{\sigma^4}{4}\right)\mathrm{Ai}\left(x_2+t+\frac{\sigma^4}{4}\right)\\ -\tfrac{1}{2}\exp\left(\frac{\sigma^6}{12}+\frac{\sigma^2x_1}{2}\right)\mathrm{Ai}\left(x_1+\frac{\sigma^4}{4}\right)\\ \times\left(1-\mathrm{e}^{\sigma^6/12}\int_{x_2}^\infty dt\,\mathrm{e}^{\sigma^2 t/2}\,\mathrm{Ai}\left(t+\frac{\sigma^4}{4}\right)\right),$$

$$G_{\mathrm{Ai},\mathbb{C},\mathrm{com}}^{\beta=1}(z_1,z_2)=-\frac{i}{2\sigma^2}\,\mathrm{sign}(y_2)(z_1-z_2^*)\,\exp\left(\frac{\sigma^6}{6}+\frac{\sigma^2(x_1+x_2)}{2}\right)\\ \times\sqrt{\mathrm{erfc}\big(|y_1|/\sigma\big)\,\mathrm{erfc}\big(|y_2|/\sigma\big)}\\ \times\int_0^\infty dt\,(\mathrm{e}^{\sigma^2 t}-1)\,\mathrm{Ai}\left(z_1+t+\frac{\sigma^4}{4}\right)\mathrm{Ai}\left(z_2^*+t+\frac{\sigma^4}{4}\right),$$

$$K^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1,z_2)=\frac{i}{2}\,\mathrm{sign}(y_2)G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1,z_2^*),$$

$$\begin{aligned}W^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1,z_2)=&\big(-2A(x_1,x_2)+B(x_1)B(x_2)+B(x_2)-B(x_1)\big)\delta^1(y_1)\delta^1(y_2)\\&+2i\,\mathrm{sign}(y_2)G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(z_2^*,x_1)\delta^1(y_1)\\&-2i\,\mathrm{sign}(y_1)G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(z_1^*,x_2)\delta^1(y_2)\\&-2i\,\mathrm{sign}(y_1)G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1^*,z_2)\\&-2i\delta^2(z_1-z_2^*)\,\mathrm{sign}(y_1)-\delta^1(y_1)\delta^1(y_2)\,\mathrm{sign}(x_2-x_1),\end{aligned}$$

$$\begin{aligned}A(x_1,x_2)=&\exp\left(\frac{\sigma^6}{6}+\frac{\sigma^2(x_1+x_2)}{2}\right)\\&\times\int_0^\infty ds\int_0^s dt\,\mathrm{e}^{\frac{1}{2}\sigma^2(s+t)}\,\mathrm{Ai}\Big(x_1+s+\frac{\sigma^4}{4}\Big)\,\mathrm{Ai}\Big(x_2+t+\frac{\sigma^4}{4}\Big),\end{aligned}$$

$$B(x)=\exp\left(\frac{\sigma^6}{12}+\frac{\sigma^2 x}{2}\right)\int_0^\infty dt\,\mathrm{e}^{\frac{1}{2}\sigma^2 t}\,\mathrm{Ai}\Big(x+t+\frac{\sigma^4}{4}\Big).\tag{2.19}$$

Clearly $G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(x_1,x_2)\to G^{\beta=1}_{\mathrm{Ai}}(x_1,x_2)$ as $\sigma\to 0$, whereas the complex part vanishes in this Hermitian limit: $G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1,z_2)\to 0$. We have also explicitly verified the corresponding limits for $K^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1,z_2)$ and $W^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1,z_2)$.

2.3. ***Limiting sine kernels on $\mathbb{R}$ and $\mathbb{C}$.*** For real eigenvalues the sine kernels are obtained by zooming into the bulk of the spectrum, sufficiently far away from the edges. The weakly non-Hermitian limit of the complex eigenvalues introduced in [Fyodorov et al. 1997a; 1997b] is taken such that

$$\sigma=N^{1/2}\sqrt{1-\tau}\tag{2.20}$$

remains finite (see Appendix B for further details). In this limit the macroscopic support of the spectral density on an ellipse shrinks to the semicircle distribution on the real axis, whereas microscopically we still have correlations of the eigenvalues in the complex plane.

The list of the known one-parameter deformations of the sine kernels for $\beta=2$ is as follows:

$\underline{\beta=2}$:

$$K^{\beta=2}_{\mathrm{sin}}(x_1,x_2)=\frac{\sin(x_1-x_2)}{\pi(x_1-x_2)}=\frac{1}{\pi}\int_0^1 dt\,\cos[(x_1-x_2)t],\tag{2.21}$$

$$K^{\beta=2}_{\mathrm{sin},\mathbb{C}}(z_1,z_2)=\frac{1}{\sigma\pi^{3/2}}\,\mathrm{e}^{-(y_1^2+y_2^2)/(2\sigma^2)}\int_0^1 dt\,\mathrm{e}^{-\sigma^2t^2}\cos[(z_1-z_2)t].\tag{2.22}$$

The corresponding spectral density of complex eigenvalues was first derived in [Fyodorov et al. 1997a] using supersymmetry, and the kernel with all correlation functions in [Fyodorov et al. 1997b; 1998] using OP; see (A.4). In the Hermitian limit $\sigma \to 0$, we have

$$K_{\sin,\mathbb{C}}^{\beta=2}(z_1, z_2) \to \sqrt{\delta^1(y_1)\delta^1(y_2)}\, K_{\sin}^{\beta=2}(x_1, x_2).$$

In [Fyodorov et al. 1998] it was shown using supersymmetric techniques that the same result holds for the microscopic density of random matrices with i.i.d. matrix elements for $\beta = 1, 2$. Further arguments in favour of universality were added in [Akemann 2002] for the kernel for $\beta = 2$ using large-N factorisation and asymptotic OP. The universal parameter is the mean macroscopic spectral density $\rho(x_0)$.

$\underline{\beta = 4}$:

$$G_{\sin}^{\beta=4}(x_1, x_2) = -\frac{\sin[2(x_1 - x_2)]}{2\pi(x_1 - x_2)},$$

$$K_{\sin}^{\beta=4}(x_1, x_2) = \frac{\partial}{\partial x_1} G_{\sin}^{\beta=4}(x_1, x_2),$$

$$W_{\sin}^{\beta=4}(x_1, x_2) = \int_0^{x_1 - x_2} dt\; G_{\sin}^{\beta=4}(t, 0) = \frac{1}{2\pi}\int_0^1 \frac{dt}{t} \sin[2(x_1 - x_2)t], \tag{2.23}$$

$$G_{\sin,\mathbb{C}}^{\beta=4}(z_1, z_2) = \frac{i2\sqrt{2}\, y_2}{\pi^{3/2}\sigma^3} \mathrm{e}^{-2y_2^2/\sigma^2} \int_0^1 \frac{dt}{t}\, \mathrm{e}^{-2\sigma^2 t^2} \sin[2(z_1 - z_2^*)t]. \tag{2.24}$$

The corresponding spectral density of complex eigenvalues was derived in [Kolesnikov and Efetov 1999] using supersymmetry, and the kernel with all correlation functions was derived in [Kanzieper 2002] using skew-OP leading to (A.9).

$\underline{\beta = 1}$:

$$G_{\sin}^{\beta=1}(x_1, x_2) = -K_{\sin}^{\beta=2}(x_1, x_2),$$

$$K_{\sin}^{\beta=1}(x_1, x_2) = \frac{\partial}{\partial x_1} G_{\sin}^{\beta=1}(x_1, x_2) = \frac{1}{\pi}\int_0^1 dt\, t \sin[(x_2 - x_1)t],$$

$$W_{\sin}^{\beta=1}(x_1, x_2) = \int_0^{x_1 - x_2} dt\; G_{\sin}^{\beta=1}(t, 0) + \tfrac{1}{2}\operatorname{sign}(x_1 - x_2), \tag{2.25}$$

$$G_{\sin,\mathbb{C},\mathrm{real}}^{\beta=1}(z_1, x_2) = -\frac{1}{\pi}\int_0^1 dt\, \mathrm{e}^{-\sigma^2 t^2} \cos[(z_1 - x_2)t],$$

$$G_{\sin,\mathbb{C},\mathrm{com}}^{\beta=1}(z_1, z_2) = -2i \operatorname{sign}(y_2) \operatorname{erfc}\left|\frac{y_1}{\sigma}\right| K_{\sin,\mathbb{C}}^{\beta=1}(z_1, z_2^*),$$