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978-1-107-07820-8 - Electricity and Magnetism for Mathematicians: A Guided Path from
Maxwell's Equations to Yang-Mills

Thomas A. Garrity

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Electricity and Magnetism for Mathematicians

This text is an introduction to some of the mathematical wonders of Maxwell's equations. These equations led to the prediction of radio waves, the realization that light is a type of electromagnetic wave, and the discovery of the special theory of relativity. In fact, almost all current descriptions of the fundamental laws of the universe can be viewed as deep generalizations of Maxwell's equations. Even more surprising is that these equations and their generalizations have led to some of the most important mathematical discoveries of the past thirty years. It seems that the mathematics behind Maxwell's equations is endless.

The goal of this book is to explain to mathematicians the underlying physics behind electricity and magnetism and to show their connections to mathematics. Starting with Maxwell's equations, the reader is led to such topics as the special theory of relativity, differential forms, quantum mechanics, manifolds, tangent bundles, connections, and curvature.

THOMAS A. GARRITY is the William R. Kenan, Jr. Professor of Mathematics at Williams, where he was the director of the Williams Project for Effective Teaching for many years. In addition to a number of research papers, he has authored or coauthored two other books, *All the Mathematics You Missed [But Need to Know for Graduate School]* and *Algebraic Geometry: A Problem Solving Approach*. Among his awards and honors is the MAA Deborah and Franklin Tepper Haimo Award for outstanding college or university teaching.

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ELECTRICITY AND MAGNETISM FOR MATHEMATICIANS

A Guided Path from Maxwell's
Equations to Yang-Mills

THOMAS A. GARRITY

Williams College, Williamstown, Massachusetts
with illustrations by Nicholas Neumann-Chun



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List of Symbols

Symbol	Name
∇	nabla
Δ	Laplacian
T	transpose
\in	element of
$O(3, \mathbb{R})$	orthogonal group
\mathbb{R}	real numbers
$\rho(\cdot, \cdot)$	Minkowski metric
$\wedge^k(\mathbb{R}^n)$	k -forms on \mathbb{R}^n
\wedge	wedge
\circ	composed with
\star	star operator
\mathcal{H}	Hilbert space
$\langle \cdot, \cdot \rangle$	inner product
\mathbb{C}	complex numbers
$L^2[0, 1]$	square integrable functions
$*$	adjoint
\subset	subset of
\mathcal{S}	Schwartz space
h	Planck constant
\cap	set intersection
\cup	set union
$GL(k, \mathbb{R})$	general linear group
C_p^∞	germ of the sheaf of differentiable functions
$\Gamma(E)$	space of all sections of E
∇	connection
\otimes	tensor product
\odot	symmetric tensor product

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