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978-1-107-07643-3 - Noise Sensitivity of Boolean: Functions and Percolation

Christophe Garban and Jeffrey E. Steif

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