Noise Sensitivity of Boolean Functions and Percolation

This is a graduate-level introduction to the theory of Boolean functions, an exciting area lying on the border of probability theory, discrete mathematics, analysis, and theoretical computer science. Certain functions are highly sensitive to noise; this can be seen via Fourier analysis on the hypercube. The key model analyzed in depth is critical percolation on the hexagonal lattice. For this model, the critical exponents, previously determined using the now-famous Schramm–Loewner evolution, appear here in the study of sensitivity behavior. Even for this relatively simple model, beyond the Fourier-analytic setup, there are three crucially important but distinct approaches: hypercontractivity of operators, connections to randomized algorithms, and viewing the spectrum as a random Cantor set. This book assumes a basic background in probability theory and integration theory. Each chapter ends with exercises, some straightforward, some challenging.

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To Laure, Victor, Camille,
Aila, Adam, and Noah

Contents

Preface

The purpose of this book is to present a self-contained theory of Boolean functions through the prism of statistical physics. The material presented here was initially designed as a set of lecture notes for the 2010 Clay summer school, and we decided to maintain the informal style which, we hope, will make this book more reader friendly.

Before going into Chapter 1, where precise definitions and statements are given, we wish to describe in an informal manner what this book is about. Our main companion through the whole book will be what one calls a **Boolean function**. This is simply a function of the following type:[1]

$$f : \{0,1\}^n \to \{0,1\}.$$

Traditionally, the study of Boolean functions arises more naturally in theoretical computer science and combinatorics. In fact, over the last 20 years, mainly thanks to the computer science community, a very rich structure has emerged concerning the properties of Boolean functions. The first part of this book (Chapters 1 to 5) is devoted to a description of some of the main achievements in this field. For example, a crucial result that has inspired much of the work presented here is the so-called KKL theorem (for Kahn–Kalai–Linial, 1989), which in essence says that any "reasonable" Boolean function has at least one variable that has a large influence on the outcome (namely at least $\Omega(\log n/n)$). See Theorem 1.14.

The second part of this book is devoted to the powerful use of Boolean functions in the context of statistical physics and in particular in percolation theory. It was recognized long ago that some of the striking properties that hold in great generality for Boolean functions have deep implications in statistical physics. For example, a version of the KKL theorem enables one to recover in an elegant manner the celebrated theorem of Kesten from

[1] In fact, in this book we view Boolean functions rather as functions from $\{-1,1\}^n \to \{-1,1\}$ because their Fourier decomposition is then simpler to write down; nevertheless this is still the same combinatorial object.

1980 that states that the critical point for percolation on $\mathbb{Z}^2$, $p_c(\mathbb{Z}^2)$, is $1/2$. More recently, Beffara and Duminil-Copin used an extension of this KKL property obtained by Graham and Grimmett to prove the conjecture that $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ for the Fortuin–Kasteleyn percolation model with parameter $q \geq 1$ in $\mathbb{Z}^2$. It is thus a remarkable fact that general principles such as the KKL property are powerful enough to capture some (not all) of the main technical difficulties that arise in understanding the phase transitions of various statistical physics models.

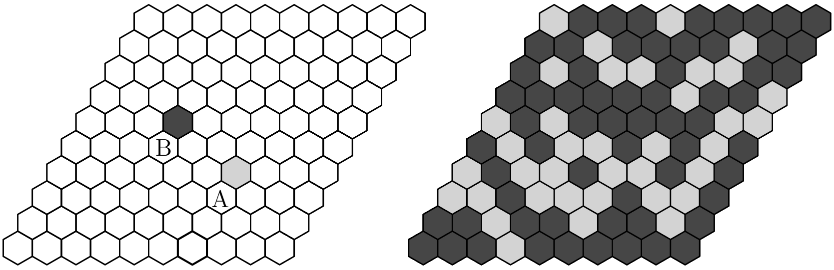
In the 1990s, Talagrand as well as Benjamini, Kalai, and Schramm pushed this connection between Boolean functions and statistical physics even further. In 1998, Benjamini, Kalai, and Schramm introduced the fruitful concept of **noise sensitivity** of Boolean functions. Their main motivation was to study the behavior of critical percolation, but let us briefly explain what noise sensitivity corresponds to in the more common situation of *voting schemes*. Suppose n voters have to decide between two candidates denoted by 0 and 1. They first have to agree on a voting procedure or *voting scheme*, which may be represented by a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$. In France or Sweden, this Boolean function would simply be the majority function on n bits whereas in the United States, the Boolean function f would be more complicated: in fancy words, it would correspond to an iterated weighted majority function on $n \approx 10^8$ voters. The collection of all votes is a certain configuration ω in the hypercube $\{0,1\}^n$. If the election is "close," it is reasonable to consider $\omega = (x_1, \ldots, x_n)$ as a uniform point chosen in $\{0,1\}^n$. In other words, we assume that each voter $i \in [n]$ tosses a fair coin in $\{0,1\}$ and votes accordingly. As such the *true* result of the election should be the output $f(\omega) = f(x_1, \ldots, x_n) \in \{0,1\}$. In reality the *actual* result of the election will rather be the output $f(\omega_\epsilon)$, where ω is an ϵ-perturbation of the configuration ω. Roughly speaking, we assume that independently for each voter $i \in [n]$, an error occurs (meaning that the value of the bit is flipped) with probability a fixed parameter $\epsilon > 0$. See Chapter 1 for precise definitions. In this language, a **noise-sensitive Boolean function** is a function for which the outputs $f(\omega)$ and $f(\omega_\epsilon)$ are almost independent of each other even with a very small level of *noise* ϵ. As an example, the *parity* function defined by $f(x_1, \ldots, x_n) = 1_{\sum x_i \equiv 1 \text{ mod } 2}$ is noise sensitive as $n \to \infty$.

As we will see, there is a very useful spectral characterization of noise sensitivity. Indeed, in the same way as a real function $g\colon \mathbb{R}\backslash\mathbb{Z} \to \mathbb{R}$ can be decomposed into $g(x) = \sum_{n\in\mathbb{Z}} \hat{g}(n) e^{2i\pi nx}$, one can decompose a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ into a Fourier–Walsh series $f(\omega) = \sum_{S\subseteq[n]} \hat{f}(S)\chi_S(\omega)$. (See Chapter 4 for details.) Noise-sensitive

Boolean functions are precisely the Boolean functions whose spectrum is concentrated on "high frequencies"; that is, most of their Fourier coefficients (in a certain analytic and quantitative manner) correspond to subsets $S \subseteq [n]$ with $|S| \gg 1$.

It is thus not surprising that with such a spectral characterization of noise sensitivity, a significant part of this book is devoted to various techniques that allow us to detect high-frequency behavior for Boolean functions. The techniques we introduce are essentially of three different flavors:

1. Analytical techniques based on **hypercontractive** estimates (Chapter 5)
2. A criterion based on randomized algorithms (Chapter 8)
3. A study of the "fractal" behavior of frequencies $S \subseteq [n]$ (Chapter 10)



To make the link with statistical physics, consider the following Boolean function, which is well known in computer science and in game theory because it represents the solution of the **Hex game**. In the figure above, we represent a Hex game on a 10×10 table: player A tries to go from the left boundary to the right using gray hexagon tiles, while player B tries to go from the top boundary to the bottom using black hexagon tiles. The players either take turns or, as in the **random turn hex game**, they toss a coin at each turn to decide who will move. At the end of the game, we obtain a tiling of the table as in the figure and the result of the game is then described by a certain Boolean function $f_{10}\colon \{A,B\}^{100} \to \{A,B\}$. Note that in the figure, player A has won. As we will see, this Boolean function (or rather the family $\{f_n\}$ defined analogously on $n \times n$ tables) is instrumental in our study of how the model of percolation responds to small random perturbations.

Boolean functions of this type are notoriously hard to study, and this book develops tools aimed at understanding such Boolean functions. In particular, we will eventually see that as $n \to \infty$, most of the Fourier transform $\hat{f}_n$ of the Hex-function f_n on an $n \times n$ table is concentrated on frequencies of size $|S|$ about $n^{3/4}$. The appearance of the surprising exponent of $3/4$ corresponds to one of the **critical exponents** that are aimed at describing the fractal geometry of critical percolation. See Chapter 2.

This high-frequency behavior of the Hex-functions $\{f_n\}$ implies readily that critical planar percolation on the triangular lattice is highly sensitive to noise (in a quantitative manner given by the above $n^{3/4}$ asymptotics). This noise sensitivity of percolation has surprising consequences concerning the model of **dynamical percolation** (see Chapter 11). We will see among other things that there exist exceptional times at which an infinite *primal* cluster coexists with an infinite *dual* cluster (Theorem 11.9), which is a very counterintuitive phenomenon in percolation theory.

In Chapter 7, we give another application to statistical physics of a very different flavor: consider the random metric R on the lattice $\mathbb{Z}^d$, $d \geq 2$, where each edge is independently declared to be of length a with probability 1/2 and of length b with probability 1/2 (where $0 < a < b < \infty$ are fixed). Fascinating conjectures have been made about the fluctuations of the random R-ball around its deterministic convex limit. In particular, it is conjectured that these fluctuations are of magnitude $R^{1/3}$ in two dimensions and that the law describing these fluctuations is intimately related to the celebrated Tracy–Widom law that describes the fluctuations of the largest eigenvalue of large random Hermitian matrices. This book certainly does not settle this stunning conjecture but it does present the best results to date on the fluctuations of this metric using a Fourier approach.

This book is structured as follows: in Chapters 1, 3, 4, 5, 8, and 9, we introduce general tools for Boolean functions that are applicable in various settings (and are thus not restricted to the context of statistical physics). Several examples in these chapters illustrate links to other active fields of mathematics. Chapter 2 is a short introduction to the model of percolation. Chapters 6, 10, and 11 are more specifically targeted toward the analysis of the **noise sensitivity of critical percolation** as well as its consequences for dynamical percolation. Chapter 7 analyzes the fluctuations of the earlier mentioned random metrics on $\mathbb{Z}^d$, $d \geq 2$. Chapter 12 explores a large variety of interesting topics tangential to the main contents of this book. Finally Chapter 13 collects some open problems.

We assume readers have the mathematical maturity of a first-year graduate student and a reasonable background in probability theory and integration theory. Having seen some percolation would be helpful but not neccesary.

Acknowledgments

This book would not be in its present shape without the help of many people whom we now wish to warmly thank. First of all, we thank Charles Newman, Vladas Sidoravicius, and Wendelin Werner for inviting us to give a course on this topic at the 2010 Clay summer school in Buzios. (This resulted in a set of lecture notes published by the Clay Institute that was the starting form of the present book.) This book would not exist without having had the opportunity to give such a course.

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Finally, at the time of concluding the writing of this book, our thoughts go out to Oded Schramm. This book would simply not exist had it not been for Oded.

Lyon and Gothenburg, October 1, 2014

Christophe Garban and Jeffrey E. Steif

Notations

Ω_n	hypercube $\{-1,1\}^n$
$\mathbf{I}_k(f)$	influence of the kth variable on f
$\mathbf{I}_k^p(f)$	influence of the kth variable on f at level p
$\mathbf{I}(f)$	total influence of the function f; see Definition 1.10
$\mathbf{Inf}(f)$	influence vector of f
$\mathbf{H}(f)$	sum of the squared influences; see Definition 5.5
$\alpha_1(R)$	probability in critical percolation to have an open path from 0 to $\partial B(0,R)$
$\alpha_1(r,R)$	multiscale version of the above
$\alpha_4(R)$	probability of a *four-arm event* from 0 to $\partial B(0,R)$
$\alpha_4(r,R)$	multiscale version of the above
χ_S	character $\chi_S(x_1,\dots,x_n) := \prod_{i\in S} x_i$
$\widehat{f}(S)$	Fourier coefficient $\hat{f}(S) = \langle f, \chi_S\rangle = \mathbb{E}[f\chi_S]$
$E_f(m), 1 \le m \le n$	*energy spectrum* of f; see Definition 4.1
$\nabla_k f$	discrete derivative along k: $\nabla_k f(\omega) := f(\omega) - f(\sigma_k(\omega))$
$\mathcal{P} = \mathcal{P}(f)$	*pivotal set* of f; see Definition 1.7
$\mathscr{S} = \mathscr{S}_f$	*spectral sample* of f; see Definition 9.1
$\hat{\mathbb{Q}}_f$	*spectral measure* of f; see Definition 9.1
$\hat{\mathbb{P}}_f$	*spectral probability measure* of f; see Definition 9.2
$f(n) \asymp g(n)$	there exists some constant $C < \infty$ such that $C^{-1} \le \frac{f(n)}{g(n)} \le C$, $\forall n \ge 1$
$f(n) \le O(g(n))$	there exists some constant $C < \infty$ such that $f(n) \le Cg(n)$, $\forall n \ge 1$
$f(n) \ge \Omega(g(n))$	there exists some constant $C > 0$ such that $f(n) \ge Cg(n)$, $\forall n \ge 1$
$f(n) = o(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$