Catalan Numbers

Catalan numbers are probably the most ubiquitous sequence of numbers in mathematics. This book provides, for the first time, a comprehensive collection of their properties and applications in combinatorics, algebra, analysis, number theory, probability theory, geometry, topology, and other areas.

After an introduction to the basic properties of Catalan numbers, the book presents 214 different kinds of objects that are counted using Catalan numbers, in the form of exercises with solutions. The reader can try solving the exercises or simply browse through them. An additional sixty-eight exercises with prescribed difficulty levels present various properties of Catalan numbers and related numbers, such as Fuss-Catalan numbers, Motzkin numbers, Schröder numbers, Narayana numbers, super Catalan numbers, q-Catalan numbers, and (q,t)-Catalan numbers. The book concludes with a history of Catalan numbers by Igor Pak and a glossary of key terms.

Whether your interest in mathematics is recreation or research, you will find plenty of fascinating and stimulating facts here.

RICHARD P. STANLEY is Professor of Applied Mathematics at the Massachusetts Institute of Technology. He is universally recognized as a leading expert in the field of combinatorics and its applications in a variety of other mathematical disciplines. He won the 2001 AMS Leroy P. Steele Prize for Mathematical Exposition for his books *Enumerative Combinatorics: Volume 1* and *Volume 2*, which contain material that form the basis for much of the present book.

Cambridge University Press
978-1-107-07509-2 - Catalan Numbers
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Frontmatter
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32 Avenue of the Americas, New York, NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107427747

First published 2015

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data
Stanley, Richard P., 1944–
Catalan numbers / Richard P. Stanley, Massachusetts Institute of Technology.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-07509-2 (hardback) – ISBN 978-1-107-42774-7 (pbk.)
1. Catalan numbers (Mathematics) 2. Combinatorial analysis.
3. Matrices. I. Title.
QA164.S74 2015
511′.6–dc23 2014043732

ISBN 978-1-107-07509-2 Hardback
ISBN 978-1-107-42774-7 Paperback

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Preface

This text had its origins in the 1970s, when I first started teaching enumerative combinatorics and became aware of the ubiquity of Catalan numbers. Originally I just made a handwritten list for my own benefit. One of the earliest such lists has survived the ravages of time and appears in Appendix A. Over the years, the list became larger and more sophisticated. When I wrote the second volume of *Enumerative Combinatorics* (published in 1999), I included sixty-six combinatorial interpretations of Catalan numbers (Exercise 6.19) as well as numerous other exercises related to Catalan numbers. Since then I have continued to collect information on Catalan numbers, posting most of it on my "Catalan addendum" web page. Now the time has come to wrap up this 40+ years of compiling Catalan material, hence the present monograph. Much of it should be accessible to mathematically talented undergraduates or even high school students, while some parts will be of interest primarily to research mathematicians.

This monograph centers on 214 combinatorial interpretations of Catalan numbers (Chapters 2 and 3). Naturally some subjectivity is involved in deciding what should count as a new interpretation. It would be easy to expand the list by several hundred more entries by a little tweaking of the current items or by "transferring bijections." For instance, there is a simple bijection φ between plane trees and ballot sequences. Thus, whenever we have a description of a Catalan object in terms of plane trees, we can apply φ and obtain a description in terms of ballot sequences. I have used my own personal tastes in deciding which such descriptions are worthwhile to include. If the reader feels that 214 is too low a number, then he or she can take solace in the solution to item 65, which discusses *infinitely* many combinatorial interpretations.

Also central to this monograph are the sixty-eight additional problems related to Catalan numbers in Chapters 4 and 5. Some of these problems deal with generalizations, refinements, and variants of Catalan numbers, namely, q-Catalan numbers, (q,t)-Catalan numbers, (a,b)-Catalan numbers, Fuss-Catalan numbers, super Catalan numbers, Narayana numbers, Motzkin numbers, and Schröder numbers. Here we have made no attempt to be as comprehensive as in Chapters 2 and 3, but we hope we have included enough information to convey the flavor of these objects.

In order to make this monograph more self-contained, we have included a chapter on basic properties of Catalan numbers (Chapter 1) and a Glossary. The Glossary defines many terms that in the text are simply given as citations to the two volumes of *Enumerative Combinatorics*. (See references [64] and [65] in the bibliography.)

The history of Catalan numbers and their ilk is quite interesting, and I am grateful to Igor Pak for contributing Appendix B on this subject. There the reader can find much fascinating information on a subject that has not hitherto received adequate attention.

Innumerable people have contributed to this monograph by sending me information on Catalan numbers. They are mentioned in the relevant places in the text. Special thanks go to David Callan, Emeric Deutsch, Igor Pak, and Lou Shapiro for their many contributions. The author was partially supported by the National Science Foundation.