Asymptotic Analysis of Random Walks

This is a companion book to *Asymptotic Analysis of Random Walks: Heavy-Tailed Distributions* by A. A. Borovkov and K. A. Borovkov. Its self-contained systematic exposition provides a highly useful resource for academic researchers and professionals interested in applications of probability in statistics, ruin theory, and queuing theory. The large deviation principle for random walks was first established by the author in 1967, under the restrictive condition that the distribution tails decay faster than exponentially. (A close assertion was proved by S. R. S. Varadhan in 1966, but only in a rather special case.) Since then, the principle has been treated in the literature only under this condition. Recently, the author, jointly with A. A. Mogul'skii, removed this restriction, finding a natural metric for which the large deviation principle for random walks holds without any conditions. This new version is presented in the book, as well as a new approach to studying large deviations in boundary crossing problems. Many results presented in the book, obtained by the author himself or jointly with co-authors, are appearing in a monograph for the first time.

A. A. Borovkov is Principal Research Fellow and a Russian Academy of Sciences Adviser at the Sobolev Institute of Mathematics, where he also served as the founding head of the Department of Probability Theory and Mathematical Statistics. He was the founding Chair in Probability Theory and Mathematical Statistics at the Novosibirsk State University in 1965. He has authored and co-authored several influential research monographs. In 1979 he was awarded (together with V. V. Sazonov and V. Statulevičius) the State Prize of the USSR for outstanding research results on asymptotic methods of probability theory, and was elected a full member of the Academy of Sciences of the USSR in 1990. He received the 2002 Russian Government Prize in Education, the 2003 Russian Academy of Sciences A. A. Markov Prize, and in 2015 the Russian Academy of Sciences A. A. Kolmogorov Prize for his joint work with A. A. Mogul'skii on the extended large deviation principle (2015).

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Asymptotic Analysis of Random Walks

Light-Tailed Distributions

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Contents

Introduction

This book is devoted mainly to the study of the asymptotic behaviour of the probabilities of rare events (large deviations) for trajectories of random walks. By random walks we mean the sequential sums of independent random variables or vectors, and also processes with independent increments. It is assumed that those random variables or vectors (jumps of random walks or increments of random processes) have distributions which are 'rapidly decreasing at infinity'. The last term means distributions which satisfy Cramér's moment condition (see below).

The book, in some sense, continues the monograph [42], where more or less the same scope of problems was considered but it was assumed that the jumps of random walks have distributions which are 'slowly decreasing at infinity', i.e. do not satisfy Cramér's condition. Such a division of the objects of study according to the speed of decrease of the distributions of jumps arises because for rapidly and slowly decreasing distributions those objects form two classes of problems which essentially differ, both in the methods of study that are required and also in the nature of the results obtained.

Each of these two classes of problems requires its own approach, to be developed, and these approaches have little in common. So, the present monograph, being a necessary addition to the book [42], hardly intersects with the latter in its methods and results. In essence, this is the second volume (after [42]) of a monograph with the single title *Asymptotic Analysis of Random Walks*.

The asymptotic analysis of random walks for rapidly decreasing distributions and, in particular, the study of the probabilities of large deviations have become one of the main subjects in modern probability theory. This can be explained as follows.

- *A random walk* is a classical object of probability theory, which presents huge theoretical interest and is a mathematical model for many important applications in mathematical statistics (sequential analysis), insurance theory (risk theory), queuing theory and many other fields.
- *Asymptotic analysis and limit theorems* (under the unbounded growth of some parameters, for example the number of random terms in a sum) form the chief

method of research in probability theory. This is due to the nature of the main laws in probability theory (they have the form of limit theorems), as well as the fact that explicit formulas or numerical values for the characteristics under investigation in particular problems, generally, do not exist and one has to find approximations for them.

- *Probabilities of large deviations* present a considerable interest from the mathematical point of view as well as in many applied problems. Finding the probabilities of large deviations allows one, for example, to find the small error probabilities in statistical hypothesis testing (error probabilities should be small), the small probabilities of the bankruptcy of insurance companies (they should be small as well), the small probabilities of the overflow of bunkers in queuing systems and so on. The so-called 'rough' theorems about the probabilities of large deviations (i.e. about the asymptotics of the logarithms of those probabilities; see Chapters 4 and 5) have found application in a series of related fields such as statistical mechanics (see, for example, [83], [89], [178]).
- *Rapidly decreasing distributions* deserve attention because the first classical results about the probabilities of large deviations of sums of random variables were obtained for rapidly decreasing distributions (i.e. distributions satisfying Cramér's condition). In many problems in mathematical statistics (especially those related to the likelihood principle), the condition of rapid decrease turns out to be automatically satisfied (see, for example, sections 6.1 and 6.2 below). A rapid (in particular, exponential) decrease in distributions often arises in queuing theory problems (for example, the Poisson order flow is widely used), in risk theory problems and in other areas. Therefore, the study of problems with rapidly decreasing jump distributions undoubtedly presents both theoretical and applied interest. Let us add that almost all the commonly used distributions in theory and applications, such as the normal distribution, the Poisson distribution, the Γ-distribution, the distribution in the Bernoulli scheme, the uniform distribution, etc. are rapidly decreasing at infinity.

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables or vectors. Put $S_0 = 0$ and

$$S_n := \sum_{k=1}^{n} \xi_k, \qquad n = 1, 2, \ldots$$

The sequence $\{S_n;\ n \geqslant 0\}$ is called a *random walk*. As has been noted, a random walk is a classical object of probability theory. Let us mention the following fundamental results related to random walks.

- *The strong law of large numbers*, on the convergence $S_n/n \underset{\text{a.s.}}{\to} \mathbf{E}\xi$ almost surely as $n \to \infty$.
- *The functional central limit theorem*, on the convergence in distribution of the process $\zeta_n(t)$, $t \in [0,1]$, with values $(S_k - A_k)/B_n$ at the points $t = k/n$, $k = 0, 1, \ldots, n$, to a stable process, where A_k, B_n are appropriate normalising constants. For example, in the case $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 = \sigma^2 < \infty$, the polygonal

line $\zeta_n(t)$ for $A_k = 0$, $B_n^2 = n\sigma^2$ converges in distribution to a standard Wiener process (the invariance principle; for details, see e.g. [11], [39] or the introduction in [42]).

- *The law of the iterated logarithm*, which establishes upper and lower bounds for the trajectories of $\{S_k\}$.

None of the above results describes the *asymptotic behaviour of the probabilities of large deviations of trajectories of* $\{S_k\}$. We mention the following main classes of problems.

(a) The study of the probabilities of large deviations of sums of random variables (or vectors); for example, the study of the asymptotics (for $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$ in the one-dimensional case) of probabilities

$$\mathbf{P}(S_n \geqslant x) \quad \text{for} \quad x \gg \sqrt{n}, \quad n \to \infty. \tag{0.0.1}$$

(b) The study of the probabilities of large deviations in boundary crossing problems. For example, in this class of problems belongs a problem concerning the asymptotics of the probabilities

$$\mathbf{P}\left(\max_{t\in[0,1]}\left(\zeta_n(t) - \frac{x}{\sqrt{n}}\,g(t)\right) \geqslant 0\right) \tag{0.0.2}$$

for an arbitrary function $g(t)$ on $[0, 1]$ in the case $x \gg \sqrt{n}$ as $n \to \infty$ (the process $\zeta_n(t)$ is defined above).

(c) The study of the more general problem about the asymptotics of the probabilities

$$\mathbf{P}\left(\zeta_n(\cdot) \in \frac{x}{\sqrt{n}}\,B\right), \qquad x \gg \sqrt{n}, \tag{0.0.3}$$

where B is an arbitrary measurable set in one or another space of the functions on $[0, 1]$.

This monograph is largely devoted to the study of problems concerning probabilities (0.0.1)–(0.0.3) and other closely related problems, under the assumption that Cramér's moment condition

[C] $\qquad \psi(\lambda) := \mathbf{E}e^{\lambda\xi} < \infty$

is satisfied *for some* $\lambda \neq 0$ (its statement here is given for a scalar ξ). This condition means (see subsection 1.1.1) that at least one of the 'tails' $\mathbf{P}(\xi \geqslant x)$ or $\mathbf{P}(\xi < -x)$ of the distribution of the variable ξ decreases as $x \to \infty$ faster than some exponent. In the monograph [42], it was assumed that condition **[C]** is not satisfied but the tails of the distribution of ξ behave in a sufficiently regular manner.

As we noted above, the first general results for problem (a) about the asymptotics of the probabilities of large deviations of sums of random variables go back to the paper of Cramér. Essential contributions to the development of this direction were also made in the papers of V.V. Petrov [149], R.R. Bahadur and R. Ranga Rao [5], C. Stone [175] and others. One should also mention the papers of B.V. Gnedenko [95], E.A. Rvacheva [163], C. Stone [174] and L.A. Shepp [166] on integro-local limit theorems for sums of random variables in the zone of normal deviations, which played an important role in extending such theorems to the zone of large deviations and forming the most adequate approach (in our opinion) to problems concerning large deviations for sums S_n. This approach is presented in Chapter 2 (see also the papers of A.A. Borovkov and A.A. Mogul'skii [56], [57], [58], [59]).

The first general results about the joint distribution of S_n and $\overline{S}_n := \max_{k\leqslant n} S_k$ (this is a particular case of the boundary crossing problem (b); see (0.0.2)) in the zone of large deviations were obtained by A. A. Borovkov in the papers [15] and [16], using analytical methods based on solving generalised Wiener–Hopf equations (in terms of Stieltjes-type integrals) for the generating function of the joint distribution of S_n and $\overline{S}_n$. The solution was obtained in the form of double transforms of the required joint distribution, expressed in terms of factorisation components of the function $1 - z\psi(\lambda)$. The double transforms, as functions of the variables λ and z, can be inverted if one knows the poles of those transforms as a function of the variable λ and applies modifications of the steepest descent methods in the variable z. Those results allowed author A.A. Borovkov in [17]–[19] to find a solution to a more general problem about the asymptotics of the probabilities (0.0.2). Later in the work by A. A. Borovkov and A. A. Mogul'skii [53], the asymptotics of the probability (0.0.2) were found for some cases using direct probability methods without the factorisation technique (see Chapter 3).

The first general results on the rough asymptotics in problem (c), i.e. on the logarithmic asymptotics of the probability of the general form (0.0.3) for arbitrary sets B in the one-dimensional case (these results constitute the large deviation principle) were obtained in the papers of S.R.S. Varadhan [177] (for a special case and $x \gg n$) and A.A. Borovkov [19] (for $x = O(n)$). In the paper of A.A. Mogul'skii [131], the results of the paper [19] were transferred to the multidimensional case. The large deviation principle was later extended to a number of other objects (see, for example, [83], [178], [179]). However, the large deviation principle in the papers [177], [19], [131] was established under a very restrictive version of Cramér's condition, that $\psi(\lambda) < \infty$ for all λ, and only for continuous trajectories $\zeta_n(t)$ with $A_k = 0$, $B_n = n$. In a more recent series of papers by A.A. Borovkov and A.A. Mogul'skii [60]–[68] (see also Chapter 4) substantial progress in the study of the large deviation principle was made: Cramér's condition, mentioned above in its strong form, was weakened or totally removed and the space of trajectories was extended up to the space of functions without discontinuities of the second kind. At the same time the form of the results changed, so it became necessary to introduce the notion of the 'extended large deviation principle' (see Chapter 4).

Let us mention that in the paper of A.A. Borovkov [19] the principle of *moderately* large deviations was established as well for $\zeta_n(t)$ as $x = o(n)$, $n \to \infty$. After that A.A. Mogul'skii [131] extended those results to the multidimensional case. In a recent paper [67] those results were strengthened (see Chapter 5 for details).

Also, note that a sizable literature has been devoted to the large deviation principle for a wide class of random processes, mainly Markov processes and processes arising in statistical mechanics (see, for example, [83], [89], [93], [178], [179]). Those publications have little in common with the present monograph in their methodology and the nature of the results obtained, so we will not touch upon them.

The above overview of the results does not in any way pretend to be complete. Our goal here is simply to identify the main milestones in the development of the limit theorems in probability theory that are presented in this monograph. A more detailed bibliography will be provided in the course of the presentation.

Let us now provide a brief description of the contents of the book.

This monograph contains main and supplemental sections. The presentation of the main sections is, generally, self-sufficient; they contain full proofs. The supplemental sections contain results which are close to the main results, but the presentation in those sections is concise and, typically, proofs are not given. We have included in these supplements results whose proofs are either too cumbersome or use methods beyond the scope of this monograph. In those cases, we supply references to the omitted proofs.

This book consists of six chapters. The first chapter contains preliminary results which will be used in what follows. In sections 1.1 and 1.2 we discuss Cramér's condition and state the properties of the *deviation function*, which plays an important role throughout the book. We obtain inequalities for the distributions of the random variables S_n and their maxima $\overline{S}_n$. In section 1.3 we establish exponential Chebyshev-type inequalities for sums of random vectors in terms of deviation functions. Section 1.5 is devoted to the integro-local limit theorems of Gnedenko and Stone.

In the second chapter we study the asymptotics of the distributions of the sums S_n of random variables and vectors (problem (a)). In section 2.1, we introduce the notion of the Cramér transform and establish the so-called reduction formula, which reduces a problem concerning integro-local theorems in the zone of large deviations to the same problem in the zone of normal deviations. In section 2.2, integro-local limit theorems in the so-called Cramér deviation zone are obtained. Section 2.3 contains supplements to the results of section 2.2, which are provided without proofs. They include local theorems for densities in the homogeneous case and integro-local limit theorems in the multidimensional case. In section 2.4, we obtain integro-local theorems on the border of the Cramér zone of deviations and in section 2.5 theorems that apply outside the Cramér zone. In the latter case, it is possible to obtain substantial results only for special classes of distributions which vary sufficiently regularly at infinity. Section 2.6 contains supplements to the results of sections 2.4 and 2.5 relating to the multidimensional case. In section 2.7, we present the large deviation principle for the sums S_n. The influence of one or several 'non-homogeneous' terms in the sum S_n on the form of the results obtained in sections 2.2–2.7 is studied in section 2.8. Section 2.9 contains additions concerning the large deviation principle for renewal functions, the probability that the sequence $\{S_n\}_{n=1}^{\infty}$ reaches a distant set and other related problems.

The third chapter is devoted to boundary crossing problems (problem (b); see (0.0.2)) for random walks. In section 3.1, we investigate the limiting behaviour of the conditional distribution of the jumps of a random walk when the end of a trajectory (the sum S_n) is fixed. This allows us to understand the probabilistic meaning of the Cramér transform. In section 3.2, conditional invariance principles and the law of the iterated logarithm are established (again, when the end of a trajectory is fixed). The problem of the crossing of an arbitrary boundary by a trajectory $\{S_k\}$ is considered in section 3.3. In section 3.4, we study the joint distribution of the first time that a random walk crosses a high level and the overshoot over this level.

The first passage time over a fixed level (in particular, the zero level) is studied in section 3.5. In section 3.6, the distribution of the first passage time over a curvilinear boundary for a class of asymptotically linear boundaries is considered.

In section 3.7, we consider the same problem as in section 3.6 but for arbitrary boundaries and normalised trajectories on the segment $[0,1]$. A generalisation of those results to the multidimensional case is provided without proof in section 3.8. In section 3.9, also without proof, we give an account of the analytical approach to the study of the joint distribution of the variables S_n and $\overline{S}_n$. The probability

$$u_{x,n}^{y} = \mathbf{P}(\overline{S}_{n-1} < x,\, S_n \geqslant x+y)$$

satisfies an integro-difference equation, so the generating function $u_x^y(z) = \sum_{n=1}^{\infty} z^n u_{x,n}^y$ satisfies the generalised (in terms of the Stieltjes integral) Wiener–Hopf integral transform on the half-line. For the Laplace transform $u^y(\lambda,z)$ of $u_x^y(z)$ (in x), we find an explicit form in terms of the so-called V-factorisation of the function $1-z\psi(\lambda)$. It turns out that this double transform can be asymptotically inverted, since it is possible to find an explicit form for the pole of the function $u^y(\lambda,z)$ in the plane λ. This allows us to find the asymptotics in x of the function $u_x^y(z)$, and then, using a modification of the steepest descent method, the asymptotics of $u_{x,n}^y$.

In the last section of Chapter 3 (section 3.10) we consider numerical methods for finding values of parameters in terms of which the investigated probabilities can be described.

Chapter 4 is devoted to the large deviation principles (l.d.p.) for trajectories $\{S_k\}_{k=1}^{n}$ (problem (c); see (0.0.3)). The first three sections are of a preliminary nature. In the first section, we introduce the notions of a local and an extended l.d.p. for random elements in an arbitrary metric space. We find conditions when a local l.d.p. implies an extended l.d.p. In the second section, we study the functional (i.e. the integral) of deviations, in terms of which all the main results of Chapters 4 and 5 are obtained. In section 4.3, we obtain exponential Chebyshev-type inequalities for trajectories of a random walk (extensions of the inequalities in section 1.3 to the case of trajectories); using these inequalities, upper bounds on the l.d.p. will be obtained. In section 4.4, we establish strong (compared to the already known) versions of the l.d.p. for continuous normalised trajectories $\zeta_n(t)$ of a random walk as $x \sim n$, $n \to \infty$. The strengthening consists in eliminating some conditions or substantially weakening them. In section 4.5, we consider an extended setting of the problem, which arises when the Cramér condition is not required to hold over the the whole axis or under simultaneous extension of the space where the trajectories are defined to the space $\mathbb{D}$ of functions without discontinuities of the second type. We introduce a metric in this space which is more relevant to the problems we consider than the Skorokhod metric, as it allows the convergence of continuous processes to discontinuous ones. In section 4.6, we establish the local l.d.p. in the space $\mathbb{D}$ and also the so-called *extended* l.d.p. In section 4.7, l.d.p.'s in the space of functions of bounded variations are presented without proof. Conditional l.d.p.'s in the space $\mathbb{D}$ of trajectories of random walks with the end that is localised (in some form) are considered in section 4.8. Some results obtained earlier in Chapter 4 are extended in section 4.9 to processes with independent increments. As a corollary, we obtain Sanov's theorem about large deviations of empirical distribution functions. In section 4.10, we briefly discuss approaches to how one can obtain l.d.p.'s for compound renewal processes and, in section 4.11, for sums of random variables defined on a finite Markov chain.

Chapter 5 is devoted to the moderately large deviation principle (m.l.d.p.) for trajectories of random walks $\zeta_n(t)$ and processes with independent increments, when $x = o(n)$, $x \gg \sqrt{n}$ as $n \to \infty$. Section 5.1 contains a presentation of the m.l.d.p. for sums S_n of random variables. The moderately large deviation principle for trajectories is formulated and proved in section 5.2. Similar results for processes with independent increments are established in section 5.3. As a corollary, we obtain a counterpart of Sanov's theorem for moderately large deviations of empirical distribution functions. The connection of the m.l.d.p. to the invariance principle is considered in section 5.4. The end of section 5.5 is devoted to conditional m.l.d.p. with a localised end.

Some applications of the results obtained in Chapters 2–5 to problems of mathematical statistics are provided in Chapter 6. The following problems are considered: the finding of parameters (the small probabilities of errors of the first and second types) for the most powerful test for two simple hypotheses (section 6.1); the finding of parameters for optimal tests in sequential analysis (also with small probabilities of errors; section 6.2).

In section 6.3, we construct asymptotically optimal non-parametric goodness of fit tests. Some asymptotic results concerning the testing of two complex parametric hypotheses are provided in section 6.4.

In section 6.5 we find asymptotics for the main characteristics in some change point problems.

Let us now mention the main distinguishing features of the book.

(1) The traditional circle of problems about limit theorems for sums S_n in the book is considerably extended. It includes the so-called boundary crossing problems related to the crossing of given boundaries by trajectories of random walks. To this circle belong, in particular, problems about probabilities of large deviations of the maxima $\overline{S}_n = \max_{k\leqslant n} S_k$ of sums of random variables, which are widely used in applications. For the first time, a systematic presentation of a unified approach to solve the above-mentioned problems is provided. In particular, we present a direct probabilistic approach to the study of boundary crossing problems for random walks.
(2) For the first time in the monographic literature, the so-called extended large deviation principle, which is valid under considerably wider assumptions than before and under a wider problem setup, is presented.
(3) For the first time in the monographic literature, the conditional l.d.p. and the moderately large deviation principle are presented.
(4) Results concerning the large deviation principle are extended to multidimensional random walks and in some cases to processes with independent increments.
(5) The book contains a considerable number of applications of the results we obtain to certain problems in mathematical statistics.

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