1

Preliminary results

1.1 Deviation function and its properties in the one-dimensional case

1.1.1 Cramér's conditions

Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables,

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} \xi_k \quad \text{for } n \geqslant 1, \quad \overline{S}_n = \max_{0 \leqslant k \leqslant n} S_k.$$

An important role in describing the distribution asymptotics of the values S_n and $\overline{S}_n$, as well as the whole trajectory of $\{S_k\}_{k=0}^{n}$ for large n, is played by the so-called *deviation function* (rate function). The deviation function is most informative if at least one of Cramér's moment conditions is met:

[C$_\pm$] *There exists* $\lambda \gtrless 0$ *such that*

$$\psi(\lambda) := \int e^{\lambda t} \mathbf{P}(\xi \in dt) < \infty.$$

A condition **[C]** will be used to mark the fulfilment of at least one of these conditions:

[C] = **[C$_+$]** $\bigcup$ **[C$_-$]**.

We denote an intersection of the conditions **[C$_\pm$]** as

[C$_0$] = **[C$_+$]** $\bigcap$ **[C$_-$]**.

Condition **[C$_0$]** means, evidently, that

$$\psi(\lambda) < \infty \quad \text{for sufficiently small } |\lambda|.$$

If

$$\lambda_+ := \sup\{\lambda : \psi(\lambda) < \infty\}, \qquad \lambda_- = \inf\{\lambda : \psi(\lambda) < \infty\},$$

then correspondingly conditions **[C$_\pm$]**, **[C]**, **[C$_0$]** can be written in the forms

$$\lambda_\pm \gtrless 0, \quad \lambda_+ - \lambda_- > 0, \quad |\lambda_\pm| > 0.$$

These conditions, which are called Cramér's conditions, characterise the decay rate of the 'tails' $F_\pm(t)$ for the distribution of a random variable ξ. When the condition $[\mathbf{C}_+]$ is met, by virtue of Chebyshev's exponential inequality we have

$$F_+(t) := \mathbf{P}(\xi \geqslant t) \leqslant e^{-\lambda t}\psi(\lambda) \quad \text{for } \lambda \in (0, \lambda_+), \quad t > 0,$$

and, therefore, $F_+(t)$ decreases exponentially as $t \to \infty$. Conversely, if $F_+(t) < ce^{-\mu t}$ for some $c < \infty$, $\mu > 0$ and for all $t > 0$, then for $\lambda \in (0, \mu)$ we have

$$\int_{-\infty}^{0} e^{\lambda t}\mathbf{P}(\xi \in dt) \leqslant 1 - F_+(0),$$

$$\int_0^\infty e^{\lambda t}\mathbf{P}(\xi \in dt) = -\int_0^\infty e^{\lambda t} dF_+(t) = F_+(0) + \lambda \int_0^\infty e^{\lambda t} F_+(t)\, dt$$
$$\leqslant F_+(0) + c\lambda \int_0^\infty e^{(\lambda-\mu)t} dt = F_+(0) + \frac{c\lambda}{\mu - \lambda} < \infty,$$

$$\psi(\lambda) \leqslant 1 + \frac{c\lambda}{\mu - \lambda} < \infty.$$

There is a similar connection between the decay rate of $F_-(t) = \mathbf{P}(\xi \leqslant -t)$ as $t \to \infty$ and the finiteness of $\psi(\lambda)$ under condition $[\mathbf{C}_-]$.

It is clear that condition $[\mathbf{C}_0]$ implies the exponential decay of $F_+(t) + F_-(t) = \mathbf{P}\big(|\xi| \geqslant t\big)$, and vice versa.

Hereafter we also use the conditions

$$[\mathbf{C}_{\infty\pm}] = \{\lambda_\pm = \pm\infty\}$$

and the condition

$$[\mathbf{C}_\infty] = \big\{|\lambda_\pm| = \infty\big\} = [\mathbf{C}_{\infty+}] \bigcap [\mathbf{C}_{\infty-}].$$

It follows from the above that the condition $[\mathbf{C}_{\infty+}]$ ($[\mathbf{C}_{\infty-}]$) is equivalent to the fact that the tail $F_+(t)$ ($F_-(t)$) diminishes faster than any exponent, as t increases.

It is clear that, for instance, an exponential distribution meets condition $[\mathbf{C}_+] \bigcap [\mathbf{C}_{\infty-}]$ while a normal distribution meets condition $[\mathbf{C}_\infty]$.

Along with Cramér's conditions we will also assume that the random variable ξ is *not degenerate*, i.e. $\xi \not\equiv$ const. (or $\mathbf{D}\xi > 0$, which is the same).

The properties of the Laplace transform $\psi(\lambda)$ of a distribution of random variable ξ are set forth in various textbooks; see e.g. [39]. Let us mention the following three properties, which we are going to use further on.

(Ψ1) *The functions* $\psi(\lambda)$ *and* $\ln\psi(\lambda)$ *are strictly convex; the ratio* $\dfrac{\psi'(\lambda)}{\psi(\lambda)}$ *strictly increases on* (λ_-, λ_+).

The analyticity property of $\psi(\lambda)$ in a strip $\operatorname{Re}\lambda \in (\lambda_-, \lambda_+)$ can be supplemented with the following 'extended' continuity property on a segment $[\lambda_-, \lambda_+]$ (on the strip $\operatorname{Re}\lambda \in [\lambda_-, \lambda_+]$).

(Ψ2) *The function* $\psi(\lambda)$ *is continuous 'from within' a segment* $[\lambda_-, \lambda_+]$*; i.e.* $\psi(\lambda_\pm \mp 0) = \psi(\lambda_\pm)$ (*the cases* $\psi(\lambda_\pm) = \infty$ *are not excluded*).

The continuity on the whole line might fail as, for instance, $\lambda_+ < \infty$, $\psi(\lambda_+) < \infty$, $\psi(\lambda_+ + 0) = \infty$, which is the case for the distribution of a random variable ξ with density $f(x) = cx^{-3}e^{-\lambda_+ x}$ for $x \geqslant 1$, $c = \text{const}$.

(Ψ3) *If* $\mathbf{E}\,|\xi|^k < \infty$ *and the right-hand side of Cramér's condition* $[\mathbf{C}_+]$ *is met, then the function* ψ *is k times right-differentiable at the point* $\lambda = 0$,

$$\psi^{(k)}(0) = \mathbf{E}\,\xi^k =: a_k,$$

and, as $\lambda \downarrow 0$,

$$\psi_\xi(\lambda) = 1 + \sum_{j=1}^{k} \frac{\lambda^j}{j!} a_j + o(\lambda^k).$$

It also follows that the next representation takes place as $\lambda \downarrow 0$:

$$\ln \psi_\xi(\lambda) = \sum_{j=1}^{k} \frac{\gamma_j \lambda^j}{j!} + o(\lambda^k), \tag{1.1.1}$$

where the γ_j are so-called *semi-invariants* (or *cumulants*) of order j of a random variable ξ. It is not difficult to check that

$$\gamma_1 = a_1, \qquad \gamma_2 = a_2^0 = \sigma^2, \qquad \gamma_3 = a_3^0, \ \ldots, \tag{1.1.2}$$

where $a_k^0 = \mathbf{E}\,(\xi - a_1)^k$ is a central moment of kth order.

1.1.2 Deviation function

Under the condition $[\mathbf{C}]$, a pivotal role in describing the asymptotics of probabilities $\mathbf{P}(S_n \geqslant x)$ is played by the *deviation function*.

Definition 1.1.1. *The deviation function of a random variable* ξ is the function [1]

$$\Lambda(\alpha) := \sup_\lambda \big(\alpha\lambda - \ln \psi(\lambda)\big). \tag{1.1.3}$$

The meaning of the name will become clear later. In classical convex analysis the right-hand side of (1.1.3) is known as the *Legendre transform* of the function $A(\lambda) := \ln \psi(\lambda)$.

Consider a function $A(\alpha, \lambda) := \alpha\lambda - A(\lambda)$ presented under the sup sign in (1.1.3). The function $-A(\lambda)$ is strictly concave (see property (Ψ1)), so the function $A(\alpha, \lambda)$ is the same (note also that $A(\alpha, \lambda) = -\ln \psi_\alpha(\lambda)$, where $\psi_\alpha(\lambda) = e^{-\lambda\alpha}\psi(\lambda)$ is the Laplace transform of the distribution of the random variable $\xi - \alpha$ and, therefore, from the 'qualitative' point of view, $A(\alpha, \lambda)$ possesses all the properties of the function $-A(\lambda)$). It follows from what has been said that there always exists a *unique* point $\lambda = \lambda(\alpha)$ on the 'extended' real line $[-\infty, \infty]$),

[1] This function is often referred to as a rate function.

where the sup in (1.1.3) is attained. When α increases, the values of $A(\alpha,\lambda)$ for $\lambda > 0$ will increase (in proportion to λ), and for $\lambda < 0$ they will decrease. Therefore, a graph of $A(\alpha,\lambda)$ as a function of λ, will, roughly speaking, 'roll over' to the right as α increases. It means that the maximum point $\lambda(\alpha)$ will also shift to the right (or will stay still, if $\lambda(\alpha) = \lambda_+$).

Let us move on to an exact formulation. The following three sets play an important role in studying the properties of the function $\Lambda(\alpha)$:

$$\mathcal{A} = \big\{\lambda : A(\lambda) < \infty\big\} = \big\{\lambda : \psi(\lambda) < \infty\big\}, \qquad \mathcal{A}' = \big\{A'(\lambda) : \lambda \in \mathcal{A}\big\},$$

and the convex envelope S of the support of the distribution of ξ. It is clear that the values $\lambda_\pm$ are bounds for the set $\mathcal{A}$. The right and left bounds $\alpha_\pm$, $s_\pm$ for the sets $\mathcal{A}'$, S, are evidently given by

$$\begin{aligned}\alpha_\pm &= A'(\lambda_\pm \mp 0) = \frac{\psi'(\lambda_\pm \mp 0)}{\psi(\lambda_\pm \mp 0)};\\ s_+ &= \sup\big\{t : \mathbf{P}(\xi \geqslant t) < 1\big\},\\ s_- &= \inf\big\{t : \mathbf{P}(\xi \leqslant t) > 0\big\},\end{aligned}$$

where $A'(\lambda_+ - 0) = \lim_{\lambda\uparrow\lambda_+} A'(\lambda)$ and $A'(\lambda_- + 0)$ is defined analogously.

If $s_+ < \infty$ (the variable ξ is bounded above), then asymptotically the function $A(\lambda)$, as $\lambda \to \infty$, will increase linearly, so that $\Lambda(\alpha) = \infty$ for $\alpha > s_+$. In a similar way $\Lambda(\alpha) = \infty$ for $s_- > -\infty$, $\alpha < s_-$. Thus, we may confine ourselves to considering the properties of the function Λ on $[s_-, s_+]$.

The value of α_+ determines the angle at which a curve $A(\lambda) = \ln\psi(\lambda)$ meets a point $\big(\lambda_+, A(\lambda_+)\big)$. The value of α_- has an analogous meaning. If $\alpha \in [\alpha_-, \alpha_+]$ then the equation $A'_\lambda(\alpha,\lambda) = 0$ or, which is the same, the equation

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = \alpha, \tag{1.1.4}$$

always has a unique solution $\lambda(\alpha)$ on the segment $[\lambda_-, \lambda_+]$ (the values of $\lambda_\pm$ can be infinite). This solution $\lambda(\alpha)$, as the inverse function to the function $\psi'(\lambda)/\psi(\lambda)$ (see (1.1.4)), which is analytic and strictly increasing on (λ_-, λ_+), is also analytic and strictly increasing on (α_-, α_+):

$$\lambda(\alpha) \uparrow \lambda_+ \quad \text{as} \quad \alpha \uparrow \alpha_+; \qquad \lambda(\alpha) \downarrow \lambda_- \quad \text{as} \quad \alpha \downarrow \alpha_-. \tag{1.1.5}$$

From the equations

$$\Lambda(\alpha) = \alpha\lambda(\alpha) - A\big(\lambda(\alpha)\big), \qquad \frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha \tag{1.1.6}$$

we obtain

$$\Lambda'(\alpha) = \lambda(\alpha) + \alpha\lambda'(\alpha) - \frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))}\,\lambda'(\alpha) = \lambda(\alpha).$$

Taking into account that $\psi'(0)/\psi(0) = a_1 = \mathbf{E}\xi$, $0 \in [\lambda_-, \lambda_+]$, $a_1 \in [\alpha_-, \alpha_+]$, we get the following representation for the function Λ:

(Λ1) *If* $\alpha_0 \in [\alpha_-, \alpha_+]$, $\alpha \in [\alpha_-, \alpha_+]$, *then*

$$\Lambda(\alpha) = \Lambda(\alpha_0) + \int_{\alpha_0}^{\alpha} \lambda(v)dv. \tag{1.1.7}$$

Since $\lambda(a_1) = \Lambda(a_1) = 0$ (*which follows from* (1.1.4) *and* (1.1.6)), *in particular for* $\alpha_0 = a_1$ *we have*

$$\Lambda(\alpha) = \int_{a_1}^{\alpha} \lambda(v)dv. \tag{1.1.8}$$

The functions $\lambda(\alpha)$, $\Lambda(\alpha)$ *are analytic on* (α_-, α_+).

Now let us consider what is happening outside the segment $[\alpha_-, \alpha_+]$. For definiteness, let $\lambda_+ > 0$. We are going to study the behaviour of the functions $\lambda(\alpha)$, $\Lambda(\alpha)$ for $\alpha \geqslant \alpha_+$. Similar considerations can be made in the case $\lambda_- < 0$, $\alpha \leqslant \alpha_-$.

First let $\lambda_+ = \infty$, i.e. let the function $\ln\psi(\lambda)$ be analytic on the whole semiaxis $\lambda > 0$, so that the tail $F_+(t)$ decreases, as $t \to \infty$, faster than any exponent. We will assume, without loss of generality, that

$$s_+ > 0, \qquad s_- < 0. \tag{1.1.9}$$

This can always be achieved using a shift transformation of a random variable; further, we assume, without loss of generality, as in many limit theorems about the distribution of S_n, that $\mathbf{E}\,\xi = 0$, using the fact that the problem of examining the distribution of S_n is 'translation-invariant'. It can also be noted that $\Lambda_{\xi-a}(\alpha-a) = \Lambda_\xi(\alpha)$ (see property (Λ4) below) and that (1.1.9) always holds, if $\mathbf{E}\,\xi = 0$.

(Λ2) (i) *If* $\lambda_+ = \infty$ *then* $\alpha_+ = s_+$.

Hence, if $\lambda_+ = \infty$, $s_+ = \infty$, then we always have $\alpha_+ = \infty$ and for any $\alpha \geqslant \alpha_-$ both (1.1.7) and (1.1.8) hold.

(ii) *If* $s_+ < \infty$ *then* $\lambda_+ = \infty$, $\alpha_+ = s_+$,

$$\Lambda(\alpha_+) = -\ln\mathbf{P}(\xi = s_+), \qquad \Lambda(\alpha) = \infty \quad \textit{for} \quad \alpha > \alpha_+.$$

Similar statements are true for s_-, α_-, λ_-.

Proof. (i) First let $s_+ < \infty$. Then the asymptotics of $\psi(\lambda)$ and $\psi'(\lambda)$, as $\lambda \to \infty$, are determined by the corresponding integrals in the vicinity of s_+:

$$\psi(\lambda) \sim \mathbf{E}\,(e^{\lambda\xi};\, \xi > s_+ - \varepsilon), \qquad \psi'(\lambda) \sim \mathbf{E}\,(\xi e^{\lambda\xi};\, \xi > s_+ - \varepsilon)$$

as $\lambda \to \infty$ and for any fixed $\varepsilon > 0$. It follows that

$$\alpha_+ = \lim_{\lambda\to\infty} \frac{\psi'(\lambda)}{\psi(\lambda)} = \lim_{\lambda\to\infty} \frac{\mathbf{E}\,(e^{\lambda\xi};\, \xi > s_+ - \varepsilon)}{\mathbf{E}\,(\xi e^{\lambda\xi};\, \xi > s_+ - \varepsilon)} = s_+.$$

If $s_+ = \infty$, then $\ln\psi(\lambda)$ increases faster than any linear function as $\lambda \to \infty$, and, therefore, the derivative $\big(\ln\psi(\lambda)\big)'$ increases without limit, $\alpha_+ = \infty$.

(ii) The first two statements are evident. Let $p_+ = \mathbf{P}(\xi = s_+) > 0$. Then

$$\psi(\lambda) \sim p_+ e^{\lambda s_+},$$
$$\alpha\lambda - \ln\psi(\lambda) = \alpha\lambda - \ln p_+ - \lambda s_+ + o(1) = (\alpha - \alpha_+)\lambda - \ln p_+ + o(1)$$

as $\lambda \to \lambda_+ = \infty$. It follows from this and (1.1.6) that

$$\Lambda(\alpha) = \begin{cases} -\ln p_+ & \text{for} \quad \alpha = \alpha_+, \\ \infty & \text{for} \quad \alpha > \alpha_+. \end{cases}$$

If $p_+ = 0$, then from the relation $\psi(\lambda) = o(e^{\lambda s_+})$ as $\lambda \to \infty$, we obtain in a similar way that $\Lambda(\alpha_+) = \infty$. The property $(\Lambda 2)$ is proved. □

Now let $0 < \lambda_+ < \infty$. Then $s_+ = \infty$. If $\alpha_+ < \infty$, then it is necessary that $\psi(\lambda_+) < \infty$, $\psi(\lambda_+ + 0) = \infty$, $\psi'(\lambda_+) < \infty$. The left derivative is meant. If $\psi(\lambda_+) = \infty$, then $\ln\psi(\lambda_+) = \infty$ and $\big(\ln\psi(\lambda)\big)' \to \infty$ as $\lambda \uparrow \lambda_+$, $\alpha_+ = \infty$, which contradicts the assumption $\alpha_+ < \infty$. Since $\psi(\lambda) = \infty$ for $\lambda > \lambda_+$, it follows that $\lambda(\alpha)$, having reached λ_+ with increasing α, stops at this point, so that for $\alpha \geqslant \alpha_+$ we have

$$\lambda(\alpha) = \lambda_+, \quad \Lambda(\alpha) = \Lambda(\alpha_+) + \lambda_+(\alpha - \alpha_+) = \alpha\lambda_+ - A(\lambda_+). \tag{1.1.10}$$

Thus, for $\alpha \geqslant \alpha_+$ the function $\lambda(\alpha)$ is a constant, and $\Lambda(\alpha)$ grows linearly. Moreover, the relations (1.1.7), (1.1.8) remain valid.

If $\alpha_+ = \infty$, then $\alpha < \alpha_+$ for all finite $\alpha \geqslant \alpha_-$, and again we are dealing with the 'regular' situation considered before (see (1.1.7), (1.1.8)). Since $\lambda(\alpha)$ is non-decreasing, those relations imply the convexity of $\Lambda(\alpha)$.

In sum, we can formulate the next property.

$(\Lambda 3)$ *The functions $\lambda(\alpha)$, $\Lambda(\alpha)$ may have discontinuities only at the points $s_\pm$ in the case $\mathbf{P}(\xi = s_\pm) > 0$. These points separate the domain (s_-, s_+) of finiteness and continuity (in the extended sense) of the function Λ from the domain $\alpha \notin [s_-, s_+]$, where $\Lambda(\alpha) = \infty$. On $[s_-, s_+]$ the function Λ is convex. (If one defines convexity in the 'extended' sense, i.e. allowing infinite values, then Λ is convex on the whole line.) The function Λ is analytic on the interval $(\alpha_-, \alpha_+) \subset (s_-, s_+)$. If $\lambda_+ < \infty$, $\alpha_+ < \infty$, then the function $\Lambda(\alpha)$ is linear on (α_+, ∞) with a slope angle of λ_+; at the boundary point α_+ the continuity of the first derivatives persists. If $\lambda_+ = \infty$, then $\Lambda(\alpha) = \infty$ on (α_+, ∞). An analogous property is valid for the function $\Lambda(\alpha)$ on $(-\infty, \alpha_-)$.*

If $\lambda_- = 0$, then $\alpha_- = a_1$ and $\lambda(\alpha) = \Lambda(\alpha) = 0$ with $\alpha \leqslant a_1$.

In fact, since $\lambda(a_1) = 0$ and $\psi(\lambda) = \infty$ for $\lambda < \lambda_- = 0 = \lambda(a_1)$, with the decrease of α down to $\alpha_- = a_1$, the point $\lambda(\alpha)$, having reached 0, stops, and $\lambda(\alpha) = 0$ for $\alpha \leqslant \alpha_- = a_1$. It follows from this and from the first identity in (1.1.6) that $\Lambda(\alpha) = 0$ for $\alpha \leqslant a_1$.

If $\lambda_- = \lambda_+ = 0$ (the condition **[C]** is not satisfied), then $\lambda(\alpha) = \Lambda(\alpha) \equiv 0$ for all α. This is evident, since the value under the sup sign in (1.1.3) is equal to $-\infty$ for all $\lambda \neq 0$. In this case the limit theorems stated in the subsequent sections would not be informative.

By summarising the properties of Λ, we can conclude, in particular, that on the whole line the function Λ is

(a) *convex: for* $\alpha, \beta \in \mathbb{R}$, $p \in [0,1]$

$$\Lambda\big(p\alpha + (1-p)\beta\big) \leqslant p\Lambda(\alpha) + (1-p)\Lambda(\beta); \tag{1.1.11}$$

(b) *lower semicontinuous*:

$$\varliminf_{\alpha\to\alpha_0} \Lambda(\alpha) \geqslant \Lambda(\alpha_0), \quad \alpha_0 \in \mathbb{R}. \tag{1.1.12}$$

Properties (a), (b) are known as general properties of the Legendre transform of a convex lower-semicontinuous function $A(\lambda)$ (see e.g. [159]).

We will also need the following properties of function Λ.

(Λ4) *Under trivial conventions about notation, for independent random variables* ξ *and* η *we have*

$$\Lambda^{(\xi+\eta)}(\alpha) = \sup_{\lambda}\big(\alpha\lambda - A^{(\xi)}(\lambda) - A^{(\eta)}(\lambda)\big) = \inf_{\gamma}\big(\Lambda^{(\xi)}(\gamma) + \Lambda^{(\eta)}(\alpha-\gamma)\big),$$

$$\Lambda^{(c\xi+b)}(\alpha) = \sup_{\lambda}\big(\alpha\lambda - \lambda b - A^{(\xi)}(\lambda c)\big) = \Lambda^{(\xi)}\left(\frac{\alpha-b}{c}\right).$$

It is clear that the infimum $\inf_\gamma$ in the first relation is reached at a point γ, such that $\lambda^{(\xi)}(\gamma) = \lambda^{(\eta)}(\alpha-\gamma)$. If ξ and η are identically distributed, then $\gamma = \alpha/2$ and, therefore,

$$\Lambda^{(\xi+\eta)}(\alpha) = \Lambda^{(\xi)}\left(\frac{\alpha}{2}\right) + \Lambda^{(\eta)}\left(\frac{\alpha}{2}\right) = 2\Lambda^{(\xi)}\left(\frac{\alpha}{2}\right).$$

It is also evident that for all $n \geqslant 2$

$$\Lambda^{(S_n)}(\alpha) = \sup\big(\alpha\lambda - nA^{(\xi)}(\lambda)\big) = n\sup\left(\frac{\alpha\lambda}{n} - A^{(\xi)}(\lambda)\right) = n\Lambda^{(\xi)}\left(\frac{\alpha}{n}\right).$$

(Λ5) *The function* $\Lambda(\alpha)$ *attains its minimal value, which is equal to* 0, *at the point* $\alpha = \mathbf{E}\,\xi = a_1$. *For definiteness, let* $\alpha_+ > 0$. *If* $a_1 = 0$, $\mathbf{E}\,|\xi^k| < \infty$, *then*

$$\lambda(0) = \Lambda(0) = \Lambda'(0) = 0, \quad \Lambda''(0) = \frac{1}{\gamma_2}, \quad \Lambda'''(0) = -\frac{\gamma_3}{\gamma_2^2}, \ \ldots \tag{1.1.13}$$

(*in the case* $\alpha_- = 0$, *right derivatives are meant*). *As* $\alpha \downarrow 0$, *the next representation takes place:*

$$\Lambda(\alpha) = \sum_{j=2}^{k} \frac{\Lambda^{(j)}(0)}{j!}\,\alpha^j + o(\alpha^k). \tag{1.1.14}$$

The semi-invariants γ_j *are defined in* (1.1.1) and (1.1.2).

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More Information

If the double-sided Cramér's condition $[\mathbf{C}_0]$ is met, then the expansion of $\Lambda(\alpha)$ into a series (1.1.14) for $k = \infty$ holds and is called a *Cramér's series*.

The proof of property $(\Lambda 5)$ should not cause any trouble, and we leave it to the reader.

$(\Lambda 6)$ *If condition* $[\mathbf{C}_+]$ *is met, then there exist constants* $c_1 > 0$ *and* c_2 *such that, for all* α,

$$\Lambda(\alpha) \geqslant c_1\alpha - c_2.$$

If $\lambda_+ = \infty$, *then* $\Lambda(\alpha) \gg \alpha$ *as* $\alpha \to \infty$, *i.e. there exists a function* $v_+(\alpha) \to \infty$, *as* $\alpha \to \infty$, *such that* $\Lambda(\alpha) \geqslant \alpha v_+(\alpha)$.

Proof. By virtue of (1.1.10), $\lambda(\alpha) \uparrow \lambda_+$ as $\alpha \uparrow \alpha_+$, so $\lambda(\alpha) = \lambda_+$ for $\alpha \geqslant \alpha_+$. Since the function Λ is convex, $\Lambda'(\alpha) = \lambda(\alpha) \uparrow$ as $\alpha \uparrow$; then, having taken a point $\alpha_0 > 0$, such that $\lambda(\alpha_0) > 0$, we obtain

$$\Lambda(\alpha) \geqslant \Lambda(\alpha_0) + \lambda(\alpha_0)(\alpha - \alpha_0) \quad \text{for all } \alpha.$$

If $\lambda_+ = \infty$, then $\lambda(\alpha) \uparrow \infty$ as $\alpha \uparrow \infty$ and

$$v_+(\alpha) := \frac{1}{\alpha}\Lambda(\alpha) = \frac{1}{\alpha}\left(\Lambda(0) + \int_0^\alpha \lambda(t)dt\right) \to \infty \quad \text{as } \alpha \to \infty.$$

The property $(\Lambda 6)$ is proved. □

If follows from this property that under condition $[\mathbf{C}_0]$ there exist constants $c_1 > 0$, $c_2 > 0$, such that

$$\Lambda(\alpha) \geqslant c_1|\alpha| - c_2 \quad \text{for all } \alpha.$$

If $|\lambda_\pm| = \infty$, then $\Lambda(\alpha) \gg |\alpha|$ as $|\alpha| \to \infty$.

$(\Lambda 7)$ *An inversion formula holds: for* $\lambda \in (\lambda_-, \lambda_+)$

$$\ln\psi(\lambda) = \sup_\alpha \big(\alpha\lambda - \Lambda(\alpha)\big). \tag{1.1.15}$$

This means that when condition $[\mathbf{C}]$ is met, the deviation function uniquely determines the Laplace transform of $\psi(\lambda)$, and, hence, the distribution of a random variable ξ. The formula (1.1.15) also indicates that the iterated Legendre transform of the convex function $\ln\psi(\lambda)$ leads to the same original function.

Proof. Let us denote the right-hand side of (1.1.15) by $T(\lambda)$ and show that $T(\lambda) = \ln\psi(\lambda)$ for $\lambda \in (\lambda_-, \lambda_+)$. If, in order to find the sup in (1.1.15), we set the derivative with respect to α of the function under the sup sign equal to zero, then we obtain the equation

$$\lambda = \Lambda'(\alpha) = \lambda(\alpha). \tag{1.1.16}$$

Since $\lambda(\alpha)$ on (α_-, α_+) is the function inverse to $\big(\ln\psi(\lambda)\big)'$ (see (1.1.4)), then for $\lambda \in (\lambda_-, \lambda_+)$ the equation (1.1.16) has an evident solution

$$\alpha = a(\lambda) := \big(\ln\psi(\lambda)\big)'. \tag{1.1.17}$$

Taking into account that $\lambda\big(a(\lambda)\big) \equiv \lambda$, we obtain

$$T(\lambda) = \lambda a(\lambda) - \Lambda\big(a(\lambda)\big),$$
$$T'(\lambda) = a(\lambda) + \lambda a'(\lambda) - \lambda\big(a(\lambda)\big)a'(\lambda) = a(\lambda).$$

Since $a(0) = a_1$ and $T(0) = -\Lambda(a_1) = 0$,

$$T(\lambda) = \int_0^\lambda a(u)du = \ln\psi(\lambda). \tag{1.1.18}$$

The statement is proved, as well as another inversion formula (the last equality in (1.1.18); this expresses $\ln\psi(\lambda)$ in terms of the integral of the function $a(\lambda)$, which is inverse to $\lambda(\alpha)$). □

By virtue of Chebyshev's exponential inequality, for all $n \geqslant 1$, $\lambda \geqslant 0$, $x \geqslant 0$, we have

$$\mathbf{P}\,(S_n \geqslant x) \leqslant e^{-\lambda x}\psi(\lambda) = \exp\big\{-\lambda x + n\ln\psi(\lambda)\big\}. \tag{1.1.19}$$

Since $\lambda(\alpha) \geqslant 0$ for $\alpha \geqslant a_1$, by setting $\alpha = x/n$ and by substituting $\lambda = \lambda(\alpha) \geqslant 0$ in (1.1.19) we obtain the property

($\Lambda 8$) *For all $n \geqslant 1$ and $\alpha = x/n \geqslant a_1$,*

$$\mathbf{P}(S_n \geqslant x) \leqslant e^{-n\Lambda(\alpha)}.$$

The next property can be named as an exponential modification of the Kolmogorov–Doob inequality.

($\Lambda 9$) **Theorem 1.1.2.** (i) *For all $n \geqslant 1$, $x \geqslant 0$ and $\lambda \geqslant 0$, one has*

$$\mathbf{P}(\overline{S}_n \geqslant x) \leqslant e^{-\lambda x}\max\big\{1, \psi^n(\lambda)\big\}. \tag{1.1.20}$$

(ii) *Let $\mathbf{E}\xi < 0$, $\lambda_1 := \max\big\{\lambda : \ \psi(\lambda) \leqslant 1\big\}$. Then, for all $n \geqslant 1$ and $x \geqslant 0$, one has*

$$\mathbf{P}\,(\overline{S}_n \geqslant x) \leqslant e^{-\lambda_1 x}. \tag{1.1.21}$$

If $\lambda_+ > \lambda_1$ then $\psi(\lambda_1) = 1$, $\Lambda(\alpha) \geqslant \lambda_1\alpha$ for all α and $\Lambda(\alpha_1) = \lambda_1\alpha_1$, where

$$\alpha_1 := \arg\big\{\lambda(\alpha) = \lambda_1\big\} = \frac{\psi'(\lambda_1)}{\psi(\lambda_1)}, \tag{1.1.22}$$

so that a line $y = \lambda_1\alpha$ is tangent to the convex function $y = \Lambda(\alpha)$ at the point $(\alpha_1, \lambda_1\alpha_1)$. In addition, along with (1.1.21), *the next inequality holds for $\alpha := x/n$:*

$$\mathbf{P}\,(\overline{S}_n \geqslant x) \leqslant e^{-n\Lambda_1(\alpha)}, \tag{1.1.23}$$

where

$$\Lambda_1(\alpha) = \begin{cases} \lambda_1\alpha & \textit{for} \quad \alpha \leqslant \alpha_1, \\ \Lambda(\alpha) & \textit{for} \quad \alpha > \alpha_1. \end{cases}$$

If $\alpha \leqslant \alpha_1$ *then the inequality* (1.1.23) *coincides with* (1.1.21)*; for* $\alpha > \alpha_1$ *it is stronger than* (1.1.21)*.*

(iii) *Let* $\mathbf{E}\xi \geqslant 0$*,* $\alpha = x/n \geqslant \mathbf{E}\xi$*. Then, for all* $n \geqslant 1$*,*

$$\mathbf{P}\,(\overline{S}_n \geqslant x) \leqslant e^{-n\Lambda(\alpha)}. \tag{1.1.24}$$

Theorem 1.1.2 distinguishes three non-overlapping possibilities,

(a) $\mathbf{E}\xi < 0$, $\lambda_+ = \lambda_1$,
(b) $\mathbf{E}\xi < 0$, $\lambda_+ > \lambda_1$,
(c) $\mathbf{E}\xi \geqslant 0$,

for which $\mathbf{P}\,(\overline{S}_n \geqslant x)$ is bounded by the right-hand sides of inequalities (1.1.21), (1.1.23), (1.1.24) correspondingly. However, by accepting some natural conventions, one can express all three stated inequalities in the unique form of (1.1.23). Indeed, let us turn to the definition (1.1.22) of α_1. As already noted (see (1.1.4)), $\lambda(\alpha)$ is a solution of the equation $\psi'(\lambda)/\psi(\lambda) = \alpha$, which is unique for

$$\alpha \in [\alpha_-, \alpha_+], \qquad \alpha_+ := \lim_{\lambda \uparrow \lambda_+} \frac{\psi'(\lambda)}{\psi(\lambda)}, \qquad \alpha_- := \lim_{\lambda \downarrow \lambda_-} \frac{\psi'(\lambda)}{\psi(\lambda)}.$$

For $\alpha \geqslant \alpha_+$ the function $\lambda(\alpha)$ is defined as a constant λ_+ (see (1.1.10)). This means that for $\lambda_1 = \lambda_+$ the value of α_1 is not uniquely defined and may take any value from α_+ to ∞, so that, by setting $\alpha_1 = \max\big\{\alpha : \lambda(\alpha) = \lambda_1 = \lambda_+\big\} = \infty$, we turn the inequality (1.1.23) in the case $\lambda_1 = \lambda_+$ (i.e. in case (a)) into the inequality

$$\mathbf{P}\,(\overline{S}_n \geqslant x) \leqslant e^{-n\lambda_1\alpha} = e^{-\lambda_1 x},$$

i.e. into the inequality (1.1.21).

If $\mathbf{E}\xi \geqslant 0$ then $\lambda_1 = 0$. If $\lambda_+ = 0$ then $\lambda_+ = \lambda_1$ and we have the same situation as before, but now $\mathbf{P}\,(\overline{S}_n \geqslant x)$ allows only the trivial bound of 1. If $\lambda_+ > 0$ then $\alpha_1 = \mathbf{E}\xi$; for $\alpha < \alpha_1$ the bound (1.1.23) is trivial again, and for $\alpha > \alpha_1$ it coincides with (1.1.24).

Corollary 1.1.3. *If we set*

$$\alpha_1 := \max\big\{\alpha : \lambda(\alpha) = \lambda_1\big\} = \begin{cases} \dfrac{\psi'(\lambda_1)}{\psi(\lambda_1)}, & \text{for} \quad \lambda_+ > \lambda_1, \\ \infty, & \text{for} \quad \lambda_+ = \lambda_1, \end{cases}$$

then the inequality (1.1.23) *holds without any additional conditions on* $\mathbf{E}\xi$ *and* λ_+ *and comprises inequalities* (1.1.21)*,* (1.1.24)*.*

From the results of [16] (where the asymptotics of the distributions of the maximum of sequential sums of random variables are studied; see also Chapter 3) it is not difficult to extract the information that the exponents in the inequality (1.1.23) are asymptotically unimprovable:

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\,(\overline{S}_n \geqslant x) = -\Lambda_1(\alpha) \quad \text{as} \quad \frac{x}{n} = \alpha;$$