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Quasi-Interpolation

MARTIN BUHMANN
Justus-Liebig-Universität, Giessen, Germany

JANIN JÄGER
Justus-Liebig-Universität, Giessen, Germany

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Dedicated to M. J. D. Powell (1936–2015) in memory of his outstanding contributions to approximation theory and for his guidance to us.

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Preface

In the range of current applications of numerical mathematics and approximation theory, it appears that there is a particular demand for techniques admitting a relatively easy formulation of approximants to data and functions with noise or that require smoothing. This is especially true when functions and data in high dimensions are to be dealt with. A conceptually easy way to formulate approximations is by interpolation, that is, data and approximant should match precisely at certain points, but this is in contrast to the demand of smoothing (for instance noisy data, which we may not want to interpolate at all), and of course there is often absolutely no guarantee that such interpolants exist, notably in more than one dimension. On the other hand, mere smoothing algorithms may not meet the accuracy and asymptotic convergence properties we wish to have.

The concept of quasi-interpolation stands in the intersection of these two demands and turns out to be exceptionally useful in a large field of applications, including the important fields of solving partial differential equations and learning, and it also has a rich and interesting mathematical theory in its own right. The idea is to mimic the concept of interpolation by replacing Lagrange functions (which may not even exist, but quasi-interpolation is almost always possible, in any dimension) with quasi-Lagrange functions that can be computed in advance. The latter need no longer match data precisely, but are local and exact on some subspaces (typically polynomials) *by design,* in contrast to the classical Lagrange functions, whose existence has to be ensured first, and only then can one look at whether they are local, for instance, or give any convergence, etc. Using those quasi-Lagrange functions one can then proceed to forming approximants in a similar way to standard interpolation, and study their properties.

It turns out that this opens the door to a rich mathematical theory on function spaces, approximation methods, their properties, and efficient and stable algorithms. At the same time, it not only justifies the use of quasi-interpolation in applications but the theory itself then provides further insight into how to develop new methods with improved properties.

In view of all this, we have tried to summarise a theory (while keeping in mind relevant applications we ourselves are keenly interested in) of quasi-interpolation in many different contexts: univariate and multivariate quasi-interpolation, periodic and non-periodic polynomials, on spheres, piecewise or not, with kernels (often radial basis function kernels), with and without shift-invariance. We have given particular emphasis to the approximation (function) spaces that are behind the quasi-interpolants (since it is crucial to choose a good space before we form the approximants themselves) and to convergence questions, but readers may of course have other preferences. For readability we summarise the references to the original literature, the history of the derivation of the results as well as references to further results in the 'Notes' section at the end of each chapter.

We hope the reader will find this collection of aspects of quasi-interpolation useful, noting several new ideas of our own, and perhaps take our contribution as a basis for further developments as well. Of course, we also hope that readers with specific scientific and engineering applications in mind can take advantage of our work on quasi-interpolation, for instance when employing some of the explicit expressions we have for quasi-Lagrange functions. No prerequisites are needed for reading this book other than mathematics-related undergraduate studies, so any graduate students (or later) of mathematics, sciences or engineering should be able to benefit from this book. We have placed an emphasis on keeping our work self-contained. In our 'Notes' sections we include information about further reading.

We could only have written this book on the basis of the enormous contributions of our colleagues, past and present, for which we are immensely grateful. In particular, our colleague Paul Sablonnière helped substantially when we began this project.

In fact, not only our book but also our earlier mathematical work, for both of us, in approximation theory depends in very many respects specifically on the concepts and theorems about quasi-interpolation that were created by other people from whom we have learned. Instead of giving a long list of names of those mathematicians, we will just mention Iso Schoenberg as one of the outstanding fathers of the approximation theory of quasi-interpolation.

We have discovered many new aspects of quasi-interpolation and approximation while writing this book; this has given us much pleasure and we hope that some of this will shine through the text. As ever, it was a pleasure to work together with our colleagues at Cambridge University Press when writing this book, and we specifically thank David Tranah for his friendship and advice. We would also like to express our gratitude to the University of Giessen for providing such an enjoyable working environment.

Due to the sheer volume of contributions from all sides, both mathematical and from practice, we cannot, of course, cover the entire theory and applications of quasi-interpolation. But we hope the reader will decide that our choices of the various parts are reasonable and provide a contribution that demonstrates the remarkable usefulness, versatility and interesting theory of quasi-interpolation.