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1

Introduction

In this chapter we will give the reader a general overview of our themes and motivations. We will give a brief outline of what functional approximations are for, especially multivariate ones, what kind of approximation methods are typical, and which ones we shall study in this book and why.

In many mathematical and scientific applications, approximation of functions of possibly very many variables (unknowns) is often needed because the theoretically known function is in fact too complicated to evaluate (especially when it would have to be evaluated very many times and/or the number of unknowns is very high) or in fact not known at all in the application except at a few points. Those could be predefined or not initially available. On top of this, the given data may be inaccurate or noisy, to a level often estimated in advance. Therefore the question arises how to approximate univariable or multivariable functions efficiently in such circumstances.

Interpolation and quasi-interpolation are both highly useful means of approximating functions and data in multivariable dimensions, say, for the notation in this book, in $\mathbf{R}^n$. They are useful methods that approximate from spaces spanned for example by polynomials, piecewise polynomials, trigonometric polynomials and exponentials or radial basis functions.

Interpolation and generally approximation using radial basis functions are very good examples since they have become a well-known and appreciated tool for approximating multivariate functions, especially when the dimensions n are really large, for which polynomial interpolation and approximation from polynomial spaces become difficult and very much depend on the geometry of the data. By contrast, the success of radial basis function approximation is linked in particular to the available variety of approximations of different kinds from the vector spaces spanned by the translates of the radial basis functions.

We point out that the most important choice at the beginning of the approximation procedure is the selection of the space of approximants.

Important features are its approximation power (highly useful convergence properties and error estimates are available, as will be seen with a special emphasis in this book), its applicability in basically any space dimension – any number of variables – and its simplicity in formulating and stating the approximation (or interpolation) problem.

Despite wishing to avoid pointwise collocation and its potential disadvantages, and therefore using quasi-interpolation in the end, much of the theoretical analysis began using interpolants from radial basis function spaces. That is, we wish to meet the approximand (the function f that is being approximated) at a number (sometimes infinite, sometimes gridded, in practice of course finite) of given points $\xi \in \mathbf{R}^n$.

Returning to the general concepts, during the past few decades quasi-interpolation has become a particularly popular approach in approximation theory, especially for smoothing purposes (e.g. when given data are noisy) and in contrast to interpolation schemes. The approach we take in this book uses a quasi-Lagrange function in linear combinations with coefficients that are derived from the approximand by linear operators, most often point evaluation, in such a way that certain low-degree polynomials are not just approximated but recovered exactly. Often this type of reproduction is only required for the leading monomial term of the approximation. On top of this, we do not always require point evaluation of approximands to form the quasi-interpolants, but also admit more general operators applied to the approximands (e.g. local integrals or derivative evaluations) before they are used to formulate the quasi-interpolating approximation.

These fundamental ideas can replace pointwise interpolation, and the approach should lead to approximation orders when sufficiently smooth general functions are approximated under suitable conditions (such as localness of the said quasi-Lagrange functions).

But now let us return for a moment specifically to the approach using radial basis functions.

We will speak about more general approaches to quasi-interpolation later on; this, however, is a good way to start explaining these ideas. This is because we note another highly important feature of radial basis functions, namely that for large well-known classes of radial basis functions the interpolation problem is uniquely solvable, sometimes with no or at most some very easily verifiable conditions on the mentioned points and their geometry, and thus completely different from the far more complicated case of multivariable polynomial interpolation, for example, where the geometry of the interpolation points is of central importance to the unique solvability of the interpolation problem. This is true, for example, for the multiquadric or inverse multiquadric radial basis function specified below, which in many applications and parts of the literature is the prime example of approximation with radial basis functions, others being inverse multiquadrics, Gaussian and Poisson kernels and thin-plate splines. We note already at this point that other than in straight polynomial interpolation, for example, the linear spaces used here to form the approximants depend on the initial points, also called 'centres' as well, because we take the radial basis functions and shift them by these centres (thus they are radially symmetric about these points, which explains their name).

Initial examples of radial basis function quasi-interpolants that will turn out to be useful are the aforementioned multiquadrics $\phi(r) = \sqrt{r^2 + c^2}$ for a real parameter c that is explicitly allowed to be zero, or thin-plate splines $\phi(r) = r^2 \log r$. The former case is an example (of which there are many) where unique interpolants are guaranteed when the points are distinct and there are at least two of them. This is the simplest conceivable condition. There are some easy linear side conditions, and conditions requiring that the data points do not lie all on a straight line, that give the same property to thin-plate splines.

So for multiquadrics and any other radial basis function that immediately gives non-singularity to the interpolation problem with finitely many distinct centres $\xi \in \Xi$ (such as inverse multiquadrics, Gaussian or Poisson kernels, or multiquadrics with $c = 0$ 'linears'; see Buhmann, 2003), we formulate the search for an interpolant of the type

$$s(x) = \sum_{\xi \in \Xi} \lambda_\xi \phi(\|x - \xi\|), \quad x \in \mathbf{R}^n,$$

by solving the linear system of equations

$$\sum_{\xi \in \Xi} \lambda_\xi \phi(\|\zeta - \xi\|) = f(\zeta), \quad \zeta \in \Xi,$$

with the interpolation matrix

$$\{\phi(\|\zeta - \xi\|)\}_{\zeta,\xi \in \Xi}.$$

In the case of extra conditions such as with thin-plate splines, this would be the search for an interpolant

$$s(x) = \sum_{\xi \in \Xi} \lambda_\xi \phi(\|x - \xi\|) + p(x), \quad x \in \mathbf{R}^n,$$

with a polynomial of degree one in the thin-plate spline case and extra conditions

$$\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0$$

for all at most linear polynomials q. Such extra conditions are not unknown for univariate splines either; even straightforward natural cubic splines

$$s(x) = \sum_{\xi\in\Xi} \lambda_\xi \phi(|x-\xi|) + cx + d, \quad x \in \mathbf{R},$$

with $\phi(r) = r^3$ enabling us to express the spline as a radial basis function, and with real constants c and d, demand side conditions for uniqueness, namely second derivatives of s vanishing at the first and last knot.

While interpolation is sometimes explicitly preferred or demanded, as it reproduces the data values at the interpolation points exactly, in many circumstances quasi-interpolation is to be preferred because it has other reproducing properties, is easier to compute, and possesses a tendency to smooth data, which is frequently desired. Quasi-interpolation forms the approximant Qf from the (data) function (approximand) f by building sums of kernel functions, which we call 'quasi-Lagrange functions' ψ, shifted as $\psi(\cdot - \xi)$ or in other ways depending on the data points $\psi_\xi \in \mathbf{R}^n$ multiplied by the function values $f(\xi)$, their derivatives, integrals or other linear operators $\lambda_\xi f$ applied to them.

The interpolation property is now replaced by other conditions, typically of ψ or ψ_ξ being from the space spanned by $\phi(\|\cdot-\xi\|)$ but restricted to locally supported or decaying functions, and that the quasi-interpolation operator be exact (reproducing) on certain function spaces, so that approximants are linear combinations of such ψ functions. For these, polynomials of a fixed maximal total degree in n unknowns is a typical example. The latter opens the door to powerful convergence results for sufficiently smooth approximands f by Taylor series or related arguments.

In summary, the difference between interpolation and quasi-interpolation is that an approximation from a finite- or infinite-dimensional linear space $\mathcal{S}$ spanned by basis functions ϕ_ξ, $\xi \in \Xi$, need not satisfy pointwise interpolation conditions such as

$$s(\zeta) = f(\zeta) \quad \text{for all } \zeta \in \Xi,$$

but the approximants are essentially from the same spaces.

We now have new means to approximate certain approximands well that are smooth enough, from certain Sobolev spaces, etc. The basis functions are no longer Lagrange functions but 'bell-shaped' quasi-Lagrange functions. Therefore, in particular, approximations can be computed without solving linear (interpolation equations) systems of equations. Moreover, it will turn out that convergence estimates are simpler to achieve with quasi-interpolation when they are employed in place of interpolation, as in particular there are no complicated estimates of operator norms (Lebesgue constants).

We have used the elements $\xi \in \Xi$, for the time being, as indices for the basis functions, but in due course it will be seen that they are inherently related to the linear spaces spanned by radial basis functions. However, giving up the explicit interpolation conditions raises the need for other conditions that fix the approximant and avoid trivial choices. In particular, we note explicitly that formulating quasi-interpolants via linear combinations

$$\sum_{\xi \in \Xi} f(\xi)\phi(\|x-\xi\|)$$

is normally useless because the radial basis functions are not at all local with respect to space (except for the special cases of Gaussian or Poisson kernels or compactly supported radial basis functions). The above approximation would provide no locality, and normally approximations to smooth functions that are locally described by Taylor expansions, for instance, will not be any good. *Spline approximations,* however, do give compact support (localness) and some polynomial recovery, but are much harder to formulate in more than one or two dimensions, partly due to their piecewise structure. We shall see that, nonetheless, at least in one dimension quasi-interpolation is related to approximation by piecewise polynomials.

In contrast to the standard approach of interpolation, incidentally, approximations created by quasi-interpolants are normally not unique, which usually has little impact on practical applications.

It is also often much more difficult to estimate the desired Lebesgue constant (operator norms) for interpolation, whereas for quasi-interpolants this is usually straightforward.

We should point out that also other approximation methods from spaces spanned by radial basis functions are possible, that is, neither interpolation nor quasi-interpolation but wavelets (Buhmann, 2003), prewavelets, compression or smoothing splines with generalised cross-validation (Wahba and Wendelberger, 1980), for example.

We address some of these ideas at the end of the book.

Having given the reasons why we consider the approximations by quasi-interpolation to be useful and interesting, we shall proceed to some general remarks, univariate approximations by quasi-interpolation, and then come to multivariable methods.

2

Generalities on Quasi-Interpolation

After we have defined quasi-interpolation and clarified the differences from ordinary interpolation (e.g. polynomial or piecewise polynomial, trigonometric or piecewise exponential, i.e. spline-based), we wish to study this new method of choice as an approximation method. This specifically includes searching for conditions on the distributions of data points, examples of a selection of spline or radial basis functions and computing approximants. It also includes showing the differences between using vector spaces created by shifts of ordinary radial basis functions and those spanned by the aforementioned quasi-Lagrange functions.

A first general form of quasi-interpolants is the following, which we will use to describe the different important parts and properties of a quasi-interpolant. This will emphasise the choices we have in the construction of quasi-interpolants.

We let $\mathcal{F}$ denote some normed linear space of functions defined on a domain Ω in $\mathbf{R}^n$ (bounded or not). For the purpose of good approximation properties, it is assumed that $\mathcal{F}$ contains a subspace $\mathcal{P}$, which we aim to reconstruct precisely via our approximation scheme. These subspaces are often linear spaces of algebraic polynomials (or trigonometric polynomials for periodic functions) that at least contain the constant functions.

For f belonging to $\mathcal{F}$, we define the quasi-interpolant Q as a *linear operator* of the type

$$Qf(x) := \sum_{\alpha \in \Lambda} \lambda_\alpha(f)\varphi_\alpha(x), \qquad x \in \Omega.$$

Here Λ is a (finite or infinite) set of indices usually closely connected to the information about the function that is available for the approximation. This typically means we use a set of data sites $\Xi \subset \Omega$, where f and φ are evaluated in some way, and then it is convenient to use Ξ itself as the index set. In other words we will use indexing $\lambda_\xi(f)\varphi_\xi$ over $\xi \in \Xi \subset \Omega$, rather than $\lambda_{\xi_\alpha}(f)\varphi_{\xi_\alpha}$ over $\alpha \in \Lambda$, and so on.

The distribution of these data sets – whether they are scattered or gridded – will play an important rôle in the further discussion.

The λ_α are continuous linear forms (*coefficient functionals*) defined on $\mathcal{F}$. In choosing the functionals we decide which information about the approximated function (f the approximand) is to be provided, for example evaluations of averages of the approximand at a small number of values or simply evaluations of the approximand. The most usual way this information is given is as point evaluations at the aforementioned data points. In this case the functionals take the form

$$\lambda_\xi(f) = f(\xi) \quad \text{for all } \xi \in \Xi.$$

But there are also other operators available that might be applied to the function (values): divided differences (particularly easy to use when the data points and centres are equally spaced and gridded), derivatives (usually of some fixed total order), or local integrals, for instance averages over balls of small radius placed at the given centres. The divided difference approach is often applied when we are *not* using predetermined linear combinations of the (radial or spline or other) basis functions but use the plain vanilla shifts themselves. In the case where $\operatorname{supp}(\varphi_\alpha)$ is bounded, the λ_α are often *finite* linear combinations of values of f at points located in a neighbourhood of this support. This property makes Q a *local operator* in the sense that the value of Qf only depends on values of f in a neighbourhood of its argument.

But now, having decided on the information available for approximation, the next decision is the choice of the function space from which these approximants (here and now: quasi-interpolants) should stem. This is determined by the φ_α, which are given functions whose properties are rather general; often they have compact support or fast decay at infinity. One can regard them as *basic functions* or *quasi-Lagrange functions* of the family of quasi-interpolants. In this work we choose spaces spanned by either polynomial or spline or radial basis functions in different ways. If we are given a set of data sites as mentioned above, the data sites are often the same as the knots in the spline case and are usually also to be employed as centres for the radial basis function spaces. In both cases it is often possible to define the function $\varphi_\xi(x) = \varphi(\|x-\xi\|)$ as a shift of just one radial or polynomial spline basis function.

Now that we have discussed the form of the quasi-interpolant, we will take a closer look at the properties a quasi-interpolant should satisfy in order to give good approximation results.

Properties of quasi-interpolants with splines, radial basis functions, etc., can be studied in L^p Banach or Hilbert spaces, especially for $p = 1$ and $p = 2$. Here, $L^p(\Omega)$, where the domain $\Omega \subset \mathbf{R}^n$ may be unbounded, is – for $1 \le p < \infty$ – the space of all real-valued Lebesgue-measurable functions with finite L^p-norm defined by its pth power

$$\|f\|_p^p = \int_\Omega |f(x)|^p \, \mathrm{d}x.$$

This is well-defined as long as this integral is finite, and for $p = \infty$ we take the supremum of $|f(x)|$ as the norm, except possibly on a subset of Ω of measure zero.

The image Qf of $f \in \mathcal{F}$, where $\mathcal{F} = L^p(\Omega)$ would be a common choice, belongs to some subspace of $\mathcal{F}$ containing the function space we want to reproduce. And also, in many cases, we wish to reproduce non-integrable functions such as polynomials, so in particular we wish the quasi-interpolant to be exact on $\mathcal{P} \ni 1$, that is,

$$Qp = p \quad \text{for all } p \in \mathcal{P}.$$

If we aim to reconstruct polynomials up to a certain degree, we naturally see that this is possible if we use the space of polynomials of at least the given order as a basis function space, or a spline space (i.e. a space of piecewise polynomials), also at least of the desired order.

For non-polynomial functions such as radial basis functions, results will show that under certain conditions on the Fourier transform of the radial basis function, a linear combination of such radial basis functions can be used to construct quasi-interpolants with polynomial-reproducing properties.

If we further assume that Q is a bounded operator, meaning that

$$\|Q\| = \sup_{f \in \mathcal{F}} \frac{\|Qf\|_{\mathcal{F}}}{\|f\|_{\mathcal{F}}} < \infty,$$

it is well known and easy to verify, by adding and subtracting an arbitrary element of the space $\mathcal{P}$ in the next equation, that for all $f \in \mathcal{F}$

$$\|f - Qf\|_{\mathcal{F}} \le (1 + \|Q\|) d(f, \mathcal{P}),$$

where

$$d(f, \mathcal{P}) = \inf\{\|f - p\| \mid p \in \mathcal{P}\}.$$

In the next section we will discuss in more detail the implications of these properties for the approximation order of the quasi-interpolant.

Quasi-interpolation is such a method devoted to multiple dimensions, as both splines and radial basis functions should, of course, be constructed for multiple dimensions. Nonetheless, the general idea is made more accessible by looking at one dimension first, because several tools in the desired construction are much the same in one as in several dimensions. And naturally, the radial basis functions themselves are, as such, one-dimensional, although taking their Fourier transforms later on, for instance, we generalise them to arbitrary space dimensions.

2.1 Approximation Properties

We required that quasi-interpolation linear operators reproduce at least all constant functions, i.e. $\mathbf{P}^0$. The majority are at least exact on linear polynomials $\mathbf{P}^1$ too, whether in one or higher dimensions. An exception to this rule is the so-called approximate approximations of Maz'ya and Schmidt (2007), where no asymptotic convergence orders (to zero) are established but errors go down only to a low threshold that is chosen *a priori*.

The next step is to consider quasi-interpolation operators that are exact on polynomials (algebraic or trigonometric) of higher degree. There are remarkable results on polynomial or rational quasi-interpolants as well as on quasi-interpolants using radial basis functions. Increasing the order of polynomial reproduction is one possible option to get better error estimates, as stated in the previous section, but these will generally be more complex and demand more information about the function.

The second option is to keep the order of reproduction fixed and increase the amount of information available about a certain bounded domain.

To give an introductory example, we assume we have decided on a quasi-interpolant that uses data on a uniform grid in one dimension. The grid centres are therefore $h\mathbf{Z}$ and we further assume that we have a Lagrange-type form of the quasi-interpolant:

$$Q_h f = \sum_{j \in \mathbf{Z}} \lambda_{hj}(f)\varphi(x - hj). \tag{2.1}$$

With decreasing h, the amount of available information on f increases as the number of basis functions used (locally) increases, so we expect a good quasi-interpolant to satisfy lower error estimates in this case. We say that a quasi-interpolant has approximation order k if at least

$$\|f - Q_h f\| \le Ch^k, \quad 0 < h < 1,$$

i.e. this is $O(h^k)$ for $h \to 0$, with an h-independent constant C.

For non-gridded data sets the mesh distance is used to derive similar results. For basis functions such as splines, which are usually locally supported, such approximation order error estimates are mostly proved using applicable forms of the Taylor expansion and using polynomial reproduction in a local form. Hence the resulting approximation order is generally equal to the order of the polynomial space to be reconstructed.

As we have already mentioned, for radial basis functions there is a bit more work to be done in order to derive quasi-interpolants that reproduce polynomials. An important result in this context states conditions on the function φ for which the shift-invariant space spanned by $\varphi(x - hj)$ contains polynomials of order k.

This is of course necessary if the quasi-interpolation operator Q_h is to reproduce these polynomials.

2.1.1 Polynomial Reproduction and Strang–Fix Conditions

We now want to turn our attention to discrete quasi-interpolants of the Schoenberg type. These were among the first quasi-interpolants introduced, and are discrete quasi-interpolants in Lagrange form

$$Qf(x) = \sum_{\xi \in \Xi} f(\xi)\varphi_\xi(x), \qquad x \in \mathbf{R},$$

for any real x. The fundamental contributions of Schoenberg are essential for almost all the work in the area of quasi-interpolation and cardinal interpolation, not only by splines but also by radial basis functions, as described in Chapter 3 and later on.

Schoenberg was the first to describe this kind of approximation for function values in $\mathbf{Z}$ (see Schoenberg 1946a, 1946b). These two papers are the foundation for most of the quasi-interpolants described in this book. Schoenberg used integer shifts of just one basis function, so his quasi-interpolant takes the form

$$Qf(x) = \sum_{j=-\infty}^{\infty} f(j)\varphi(x-j), \qquad x \in \mathbf{R}.$$

The key goal was to find conditions on the function φ that is local (e.g. decaying quickly, perhaps exponentially, or locally supported), which enforce Qf to reproduce polynomials and possibly to interpolate any function f on $\mathbf{Z}$.

In the original literature, Schoenberg referred to this kind of approximation as smoothing interpolation, but it is exactly what we call quasi-interpolation today. We will see many applications of the next theorem and its enhancements, but we are going to state the proof of the simplest versions so that the essence of the proof becomes clear.

The basic tool of Schoenberg's approximation is that of Fourier transforms in one and more dimensions,

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, \mathrm{e}^{-\mathrm{i}\xi\cdot x}\, \mathrm{d}x, \qquad \xi \in \mathbf{R}^n,$$

and the Poisson summation formula. The Poisson summation formula taken from Stein and Weiss (1971) states that for any Schwartz test function $f \in \mathcal{S}$ with Fourier transform $\hat{f}$, if we consider its periodisation

$$F(x) = \sum_{\ell \in \mathbf{Z}^n} f(x + 2\pi\ell), \qquad x \in \mathbf{R}^n,$$