## Contents

*Preface to the Second Edition*  
Page xiii  
*Preface to the First Edition*  
Page xv

### 1 Background

1.1 The Gamma and Beta Functions  
1.2 Hypergeometric Series  
1.2.1 Lauricella series  
1.3 Orthogonal Polynomials of One Variable  
1.3.1 General properties  
1.3.2 Three-term recurrence  
1.4 Classical Orthogonal Polynomials  
1.4.1 Hermite polynomials  
1.4.2 Laguerre polynomials  
1.4.3 Gegenbauer polynomials  
1.4.4 Jacobi polynomials  
1.5 Modified Classical Polynomials  
1.5.1 Generalized Hermite polynomials  
1.5.2 Generalized Gegenbauer polynomials  
1.5.3 A limiting relation  
1.6 Notes  

### 2 Orthogonal Polynomials in Two Variables

2.1 Introduction  
2.2 Product Orthogonal Polynomials  
2.3 Orthogonal Polynomials on the Unit Disk  
2.4 Orthogonal Polynomials on the Triangle  
2.5 Orthogonal Polynomials and Differential Equations  
2.6 Generating Orthogonal Polynomials of Two Variables  
2.6.1 A method for generating orthogonal polynomials
### Contents

2.6.2 Orthogonal polynomials for a radial weight 40  
2.6.3 Orthogonal polynomials in complex variables 41  
2.7 First Family of Koornwinder Polynomials 45  
2.8 A Related Family of Orthogonal Polynomials 48  
2.9 Second Family of Koornwinder Polynomials 50  
2.10 Notes 54

3 General Properties of Orthogonal Polynomials in Several Variables 57  
3.1 Notation and Preliminaries 58  
3.2 Moment Functionals and Orthogonal Polynomials in Several Variables 60  
3.2.1 Definition of orthogonal polynomials 60  
3.2.2 Orthogonal polynomials and moment matrices 64  
3.2.3 The moment problem 67  
3.3 The Three-Term Relation 70  
3.3.1 Definition and basic properties 70  
3.3.2 Favard’s theorem 73  
3.3.3 Centrally symmetric integrals 76  
3.3.4 Examples 79  
3.4 Jacobi Matrices and Commuting Operators 82  
3.5 Further Properties of the Three-Term Relation 87  
3.5.1 Recurrence formula 87  
3.5.2 General solutions of the three-term relation 94  
3.6 Reproducing Kernels and Fourier Orthogonal Series 96  
3.6.1 Reproducing kernels 97  
3.6.2 Fourier orthogonal series 101  
3.7 Common Zeros of Orthogonal Polynomials in Several Variables 103  
3.8 Gaussian Cubature Formulae 107  
3.9 Notes 112

4 Orthogonal Polynomials on the Unit Sphere 114  
4.1 Spherical Harmonics 114  
4.2 Orthogonal Structures on $S^d$ and on $B^d$ 119  
4.3 Orthogonal Structures on $B^d$ and on $S^{d+m-1}$ 125  
4.4 Orthogonal Structures on the Simplex 129  
4.5 Van der Corput–Schaake Inequality 133  
4.6 Notes 136

5 Examples of Orthogonal Polynomials in Several Variables 137  
5.1 Orthogonal Polynomials for Simple Weight Functions 137  
5.1.1 Product weight functions 138  
5.1.2 Rotation-invariant weight functions 138
5.1 Classical Orthogonal Polynomials on \( \mathbb{R}^d \)

5.1.3 Multiple Hermite polynomials on \( \mathbb{R}^d \)  
5.1.4 Multiple Laguerre polynomials on \( \mathbb{R}^*_d \)  

5.2 Classical Orthogonal Polynomials on the Unit Ball

5.2.1 Orthonormal bases  
5.2.2 Appell’s monic orthogonal and biorthogonal polynomials  
5.2.3 Reproducing kernel with respect to \( W_B^\mu \) on \( B^d \)  

5.3 Classical Orthogonal Polynomials on the Simplex  

5.4 Orthogonal Polynomials via Symmetric Functions

5.4.1 Two general families of orthogonal polynomials  
5.4.2 Common zeros and Gaussian cubature formulae  

5.5 Chebyshev Polynomials of Type \( A_d \)  

5.6 Sobolev Orthogonal Polynomials on the Unit Ball

5.6.1 Sobolev orthogonal polynomials defined via the gradient operator  
5.6.2 Sobolev orthogonal polynomials defined via the Laplacian operator  

5.7 Notes  

6 Root Systems and Coxeter Groups

6.1 Introduction and Overview  
6.2 Root Systems

6.2.1 Type \( A_{d-1} \)  
6.2.2 Type \( B_d \)  
6.2.3 Type \( I_2(m) \)  
6.2.4 Type \( D_d \)  
6.2.5 Type \( H_3 \)  
6.2.6 Type \( F_4 \)  
6.2.7 Other types  
6.2.8 Miscellaneous results  

6.3 Invariant Polynomials

6.3.1 Type \( A_{d-1} \) invariants  
6.3.2 Type \( B_d \) invariants  
6.3.3 Type \( D_d \) invariants  
6.3.4 Type \( I_2(m) \) invariants  
6.3.5 Type \( H_3 \) invariants  
6.3.6 Type \( F_4 \) invariants  

6.4 Differential–Difference Operators  
6.5 The Intertwining Operator  
6.6 The \( \kappa \)-Analogue of the Exponential  
6.7 Invariant Differential Operators  
6.8 Notes
## Contents

### 7 Spherical Harmonics Associated with Reflection Groups

- **7.1 h-Harmonic Polynomials** 208
- **7.2 Inner Products on Polynomials** 217
- **7.3 Reproducing Kernels and the Poisson Kernel** 221
- **7.4 Integration of the Intertwining Operator** 224
- **7.5 Example: Abelian Group $\mathbb{Z}_d^2$** 228
  - **7.5.1 Orthogonal basis for h-harmonics** 228
  - **7.5.2 Intertwining and projection operators** 232
  - **7.5.3 Monic orthogonal basis** 235
- **7.6 Example: Dihedral Groups** 240
  - **7.6.1 An orthonormal basis of $\mathcal{H}_n(h^2_{\alpha,\beta})$** 241
  - **7.6.2 Cauchy and Poisson kernels** 248
- **7.7 The Dunkl Transform** 250
- **7.8 Notes** 256

### 8 Generalized Classical Orthogonal Polynomials

- **8.1 Generalized Classical Orthogonal Polynomials on the Ball** 258
  - **8.1.1 Definition and differential–difference equations** 258
  - **8.1.2 Orthogonal basis and reproducing kernel** 263
  - **8.1.3 Orthogonal polynomials for $\mathbb{Z}_d^2$-invariant weight functions** 266
  - **8.1.4 Reproducing kernel for $\mathbb{Z}_d^2$-invariant weight functions** 268
- **8.2 Generalized Classical Orthogonal Polynomials on the Simplex** 271
  - **8.2.1 Weight function and differential–difference equation** 271
  - **8.2.2 Orthogonal basis and reproducing kernel** 273
  - **8.2.3 Monic orthogonal polynomials** 276
- **8.3 Generalized Hermite Polynomials** 278
- **8.4 Generalized Laguerre Polynomials** 283
- **8.5 Notes** 287

### 9 Summability of Orthogonal Expansions

- **9.1 General Results on Orthogonal Expansions** 289
  - **9.1.1 Uniform convergence of partial sums** 289
  - **9.1.2 Cesàro means of the orthogonal expansion** 293
- **9.2 Orthogonal Expansion on the Sphere** 296
- **9.3 Orthogonal Expansion on the Ball** 299
- **9.4 Orthogonal Expansion on the Simplex** 304
- **9.5 Orthogonal Expansion of Laguerre and Hermite Polynomials** 306
- **9.6 Multiple Jacobi Expansion** 311
- **9.7 Notes** 315
# Contents

## 10 Orthogonal Polynomials Associated with Symmetric Groups

10.1 Partitions, Compositions and Orderings 318
10.2 Commuting Self-Adjoint Operators 320
10.3 The Dual Polynomial Basis 322
10.4 $S_d$-Invariant Subspaces 329
10.5 Degree-Changing Recurrences 334
10.6 Norm Formulae 337
  10.6.1 Hook-length products and the pairing norm 337
  10.6.2 The biorthogonal-type norm 341
  10.6.3 The torus inner product 343
  10.6.4 Monic polynomials 346
  10.6.5 Normalizing constants 346
10.7 Symmetric Functions and Jack Polynomials 350
10.8 Miscellaneous Topics 357
10.9 Notes 362

## 11 Orthogonal Polynomials Associated with Octahedral Groups, and Applications

11.1 Introduction 364
11.2 Operators of Type $B$ 365
11.3 Polynomial Eigenfunctions of Type $B$ 368
11.4 Generalized Binomial Coefficients 376
11.5 Hermite Polynomials of Type $B$ 383
11.6 Calogero–Sutherland Systems 385
  11.6.1 The simple harmonic oscillator 386
  11.6.2 Root systems and the Laplacian 387
  11.6.3 Type A models on the line 387
  11.6.4 Type A models on the circle 389
  11.6.5 Type B models on the line 392
11.7 Notes 394

References 396
Author Index 413
Symbol Index 416
Subject Index 418