Contents

Preface to the Second Edition *page* xiii
Preface to the First Edition xv

1 Background 1
1.1 The Gamma and Beta Functions 1
1.2 Hypergeometric Series 3
1.2.1 Lauricella series 5
1.3 Orthogonal Polynomials of One Variable 6
1.3.1 General properties 6
1.3.2 Three-term recurrence 9
1.4 Classical Orthogonal Polynomials 13
1.4.1 Hermite polynomials 13
1.4.2 Laguerre polynomials 14
1.4.3 Gegenbauer polynomials 16
1.4.4 Jacobi polynomials 20
1.5 Modified Classical Polynomials 22
1.5.1 Generalized Hermite polynomials 24
1.5.2 Generalized Gegenbauer polynomials 25
1.5.3 A limiting relation 27
1.6 Notes 27

2 Orthogonal Polynomials in Two Variables 28
2.1 Introduction 28
2.2 Product Orthogonal Polynomials 29
2.3 Orthogonal Polynomials on the Unit Disk 30
2.4 Orthogonal Polynomials on the Triangle 35
2.5 Orthogonal Polynomials and Differential Equations 37
2.6 Generating Orthogonal Polynomials of Two Variables 38
2.6.1 A method for generating orthogonal polynomials 38

2.6.2 Orthogonal polynomials for a radial weight 40
2.6.3 Orthogonal polynomials in complex variables 41
2.7 First Family of Koornwinder Polynomials 45
2.8 A Related Family of Orthogonal Polynomials 48
2.9 Second Family of Koornwinder Polynomials 50
2.10 Notes 54

3 General Properties of Orthogonal Polynomials in Several Variables 57
3.1 Notation and Preliminaries 58
3.2 Moment Functionals and Orthogonal Polynomials in Several Variables 60
3.2.1 Definition of orthogonal polynomials 60
3.2.2 Orthogonal polynomials and moment matrices 64
3.2.3 The moment problem 67
3.3 The Three-Term Relation 70
3.3.1 Definition and basic properties 70
3.3.2 Favard's theorem 73
3.3.3 Centrally symmetric integrals 76
3.3.4 Examples 79
3.4 Jacobi Matrices and Commuting Operators 82
3.5 Further Properties of the Three-Term Relation 87
3.5.1 Recurrence formula 87
3.5.2 General solutions of the three-term relation 94
3.6 Reproducing Kernels and Fourier Orthogonal Series 96
3.6.1 Reproducing kernels 97
3.6.2 Fourier orthogonal series 101
3.7 Common Zeros of Orthogonal Polynomials in Several Variables 103
3.8 Gaussian Cubature Formulae 107
3.9 Notes 112

4 Orthogonal Polynomials on the Unit Sphere 114
4.1 Spherical Harmonics 114
4.2 Orthogonal Structures on S^d and on B^d 119
4.3 Orthogonal Structures on B^d and on S^{d+m-1} 125
4.4 Orthogonal Structures on the Simplex 129
4.5 Van der Corput–Schaake Inequality 133
4.6 Notes 136

5 Examples of Orthogonal Polynomials in Several Variables 137
5.1 Orthogonal Polynomials for Simple Weight Functions 137
5.1.1 Product weight functions 138
5.1.2 Rotation-invariant weight functions 138

5.1.3 Multiple Hermite polynomials on $\mathbb{R}^d$ 139
5.1.4 Multiple Laguerre polynomials on $\mathbb{R}^d_+$ 141
5.2 Classical Orthogonal Polynomials on the Unit Ball 141
5.2.1 Orthonormal bases 142
5.2.2 Appell's monic orthogonal and biorthogonal polynomials 143
5.2.3 Reproducing kernel with respect to W^B_μ on B^d 148
5.3 Classical Orthogonal Polynomials on the Simplex 150
5.4 Orthogonal Polynomials via Symmetric Functions 154
5.4.1 Two general families of orthogonal polynomials 154
5.4.2 Common zeros and Gaussian cubature formulae 156
5.5 Chebyshev Polynomials of Type $\mathscr{A}_d$ 159
5.6 Sobolev Orthogonal Polynomials on the Unit Ball 165
5.6.1 Sobolev orthogonal polynomials defined via the gradient operator 165
5.6.2 Sobolev orthogonal polynomials defined via the Laplacian operator 168
5.7 Notes 171

6 Root Systems and Coxeter Groups 174
6.1 Introduction and Overview 174
6.2 Root Systems 176
6.2.1 Type A_{d-1} 179
6.2.2 Type B_d 179
6.2.3 Type $I_2(m)$ 180
6.2.4 Type D_d 181
6.2.5 Type H_3 181
6.2.6 Type F_4 182
6.2.7 Other types 182
6.2.8 Miscellaneous results 182
6.3 Invariant Polynomials 183
6.3.1 Type A_{d-1} invariants 185
6.3.2 Type B_d invariants 186
6.3.3 Type D_d invariants 186
6.3.4 Type $I_2(m)$ invariants 186
6.3.5 Type H_3 invariants 186
6.3.6 Type F_4 invariants 187
6.4 Differential–Difference Operators 187
6.5 The Intertwining Operator 192
6.6 The κ-Analogue of the Exponential 200
6.7 Invariant Differential Operators 202
6.8 Notes 207

7 Spherical Harmonics Associated with Reflection Groups 208
7.1 ***h***-Harmonic Polynomials 208
7.2 Inner Products on Polynomials 217
7.3 Reproducing Kernels and the Poisson Kernel 221
7.4 Integration of the Intertwining Operator 224
7.5 Example: Abelian Group $\mathbb{Z}_2^d$ 228
7.5.1 Orthogonal basis for h-harmonics 228
7.5.2 Intertwining and projection operators 232
7.5.3 Monic orthogonal basis 235
7.6 Example: Dihedral Groups 240
7.6.1 An orthonormal basis of $\mathscr{H}_n(h^2_{\alpha,\beta})$ 241
7.6.2 Cauchy and Poisson kernels 248
7.7 The Dunkl Transform 250
7.8 Notes 256

8 Generalized Classical Orthogonal Polynomials 258
8.1 Generalized Classical Orthogonal Polynomials on the Ball 258
8.1.1 Definition and differential–difference equations 258
8.1.2 Orthogonal basis and reproducing kernel 263
8.1.3 Orthogonal polynomials for $\mathbb{Z}_2^d$-invariant weight functions 266
8.1.4 Reproducing kernel for $\mathbb{Z}_2^d$-invariant weight functions 268
8.2 Generalized Classical Orthogonal Polynomials on the Simplex 271
8.2.1 Weight function and differential–difference equation 271
8.2.2 Orthogonal basis and reproducing kernel 273
8.2.3 Monic orthogonal polynomials 276
8.3 Generalized Hermite Polynomials 278
8.4 Generalized Laguerre Polynomials 283
8.5 Notes 287

9 Summability of Orthogonal Expansions 289
9.1 General Results on Orthogonal Expansions 289
9.1.1 Uniform convergence of partial sums 289
9.1.2 Cesàro means of the orthogonal expansion 293
9.2 Orthogonal Expansion on the Sphere 296
9.3 Orthogonal Expansion on the Ball 299
9.4 Orthogonal Expansion on the Simplex 304
9.5 Orthogonal Expansion of Laguerre and Hermite Polynomials 306
9.6 Multiple Jacobi Expansion 311
9.7 Notes 315

10 Orthogonal Polynomials Associated with Symmetric Groups 318
10.1 Partitions, Compositions and Orderings 318
10.2 Commuting Self-Adjoint Operators 320
10.3 The Dual Polynomial Basis 322
10.4 S_d-Invariant Subspaces 329
10.5 Degree-Changing Recurrences 334
10.6 Norm Formulae 337
10.6.1 Hook-length products and the pairing norm 337
10.6.2 The biorthogonal-type norm 341
10.6.3 The torus inner product 343
10.6.4 Monic polynomials 346
10.6.5 Normalizing constants 346
10.7 Symmetric Functions and Jack Polynomials 350
10.8 Miscellaneous Topics 357
10.9 Notes 362

11 Orthogonal Polynomials Associated with Octahedral Groups, and Applications 364
11.1 Introduction 364
11.2 Operators of Type B 365
11.3 Polynomial Eigenfunctions of Type B 368
11.4 Generalized Binomial Coefficients 376
11.5 Hermite Polynomials of Type B 383
11.6 Calogero–Sutherland Systems 385
11.6.1 The simple harmonic oscillator 386
11.6.2 Root systems and the Laplacian 387
11.6.3 Type A models on the line 387
11.6.4 Type A models on the circle 389
11.6.5 Type B models on the line 392
11.7 Notes 394

References 396
Author Index 413
Symbol Index 416
Subject Index 418