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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Orthogonal Polynomials of Several Variables

Second Edition

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To our wives

Philomena and Litian

with deep appreciation

Contents

Preface to the Second Edition

In this second edition, several major changes have been made to the structure of the book. A new chapter on orthogonal polynomials in two variables has been added to provide a more convenient source of information for readers concerned with this topic. The chapter collects results previously scattered in the book, specializing results in several variables to two variables whenever necessary, and incorporates further results not covered in the first edition. We have also added a new chapter on orthogonal polynomials on the unit sphere, which consolidates relevant results in the first edition and adds further results on the topic. Since the publication of the first edition in 2001, considerable progress has been made in this research area. We have incorporated several new developments, updated the references and, accordingly, edited the notes at the ends of relevant chapters. In particular, Chapter 5, "Examples of Orthogonal Polynomials in Several Variables", has been completely rewritten and substantially expanded. New materials have also been added to several other chapters. An index of symbols is given at the end of the book.

Another change worth mentioning is that orthogonal polynomials have been renormalized. Some families of orthogonal polynomials in several variables have expressions in terms of classical orthogonal polynomials in one variable. To provide neater expressions without constants in square roots they are now given in the form of orthogonal rather than orthonormal polynomials as in the first edition. The L^2 norms have been recomputed accordingly.

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Charles F. Dunkl
Yuan Xu

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Charles F. Dunkl and Yuan Xu
Frontmatter
More information

Preface to the First Edition

The study of orthogonal polynomials of several variables goes back at least as far as Hermite. There have been only a few books on the subject since: Appell and de Fériet [1926] and Erdélyi *et al.* [1953]. Twenty-five years have gone by since Koornwinder's survey article [1975]. A number of individuals who need techniques from this topic have approached us and suggested (even asked) that we write a book accessible to a general mathematical audience.

It is our goal to present the developments of very recent research to a readership trained in classical analysis. We include applied mathematicians and physicists, and even chemists and mathematical biologists, in this category.

While there is some material about the general theory, the emphasis is on classical types, by which we mean families of polynomials whose weight functions are supported on standard domains such as the simplex and the ball, or Gaussian types, which satisfy differential–difference equations and for which fairly explicit formulae exist. The term "difference" refers to operators associated with reflections in hyperplanes. The most desirable situation occurs when there is a set of commuting self-adjoint operators whose simultaneous eigenfunctions form an orthogonal basis of polynomials. As will be seen, this is still an open area of research for some families.

With the intention of making this book useful to a wide audience, for both reference and instruction, we use familiar and standard notation for the analysis on Euclidean space and assume a basic knowledge of Fourier and functional analysis, matrix theory and elementary group theory. We have been influenced by the important books of Bailey [1935], Szegő [1975] and Lebedev [1972] in style and taste.

Here is an overview of the contents. Chapter 1 is a summary of the key one-variable methods and definitions: gamma and beta functions, the classical and related orthogonal polynomials and their structure constants, and hypergeometric and Lauricella series. The multivariable analysis begins in Chapter 2 with some examples of orthogonal polynomials and spherical harmonics and specific two-variable examples such as Jacobi polynomials on various domains and disk polynomials. There is a discussion of the moment problem, general properties of orthogonal polynomials of several variables and matrix three-term recurrences in Chapter 3. Coxeter groups are treated systematically in a self-contained way, in a style suitable for the analyst, in Chapter 4 (a knowledge of representation theory is not necessary). The chapter goes on to introduce differential–difference operators, the intertwining operator and the analogue of the exponential function and concludes with the construction of invariant differential operators. Chapter 5 is a presentation of h-harmonics, the analogue of harmonic homogeneous polynomials associated with reflection groups; there are some examples of specific reflection groups as well as an application to proving the isometric properties of the generalized Fourier transform. This transform uses an analogue of the exponential function. It contains the classical Hankel transform as a special case. Chapter 6 is a detailed treatment of orthogonal polynomials on the simplex, the ball and of Hermite type. Then, summability theorems for expansions in terms of these polynomials are presented in Chapter 7; the main method is Cesàro (C, δ) summation, and there are precise results on which values of δ give positive or bounded linear operators. Nonsymmetric Jack polynomials appear in Chapter 8; this chapter contains all necessary details for their derivation, formulae for norms, hook-length products and computations of the structure constants. There is a proof of the Macdonald–Mehta–Selberg integral formula. Finally, Chapter 9 shows how to use the nonsymmetric Jack polynomials to produce bases associated with the octahedral groups. This chapter has a short discussion of how these polynomials and related operators are used to solve the Schrödinger equations of Calogero–Sutherland systems; these are exactly solvable models of quantum mechanics involving identical particles in a one-dimensional space. Both Chapters 8 and 9 discuss orthogonal polynomials on the torus and of Hermite type.

The bibliography is intended to be reasonably comprehensive into the near past; the reader is referred to Erdélyi *et al.* [1953] for older papers, and Internet databases for the newest articles. There are occasions in the book where we suggest some algorithms for possible symbolic algebra use; the reader is encouraged to implement them in his/her favorite computer algebra system but again the reader is referred to the Internet for specific published software.

There are several areas of related current research that we have deliberately avoided: the role of special functions in the representation theory of Lie groups (see Dieudonné [1980], Hua [1963], Vilenkin [1968], Vilenkin and Klimyk [1991a, b, c, 1995]), basic hypergeometric series and orthogonal polynomials of q-type (see Gasper and Rahman [1990], Andrews, Askey and Roy [1999]), quantum groups (Koornwinder [1992], Noumi [1996], Koelink [1996] and Stokman [1997]), Macdonald symmetric polynomials (a generalization of the q-type) (see Macdonald [1995, 1998]). These topics touch on algebra, combinatorics and analysis; and some classical results can be obtained as limiting cases for $q \to 1$. Nevertheless, the material in this book can stand alone and 'q' is not needed in the proofs.

We gratefully acknowledge support from the National Science Foundation over the years for our original research, some of which is described in this book. Also we are grateful to the mathematics departments of the University of Oregon for granting sabbatical leave and the University of Virginia for inviting Y. X. to visit for a year, which provided the opportunity for this collaboration.

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Yuan Xu