Introduction and Summary

It is widely known that the commutative probability theory based on Kolmogorov's measure theoretic axioms and the quantum probability theory based on von Neumann's postulates were both created at about the same time in the 1930s (see, e.g., Kolmogorov [Kol50] and von Neumann [vNeu55] for the origins of these 2 very different but parallel lines of theory). Following Kolmogorov's original work, theory of the classical Markov processes has been the subject of intensive research for the last few decades. The classical theory, its applications, and their connections to other areas of research have been systematically developed by many prominent probabilists as illuminated by their widely referenced monographs such as those of Dynkin [Dyn62], Blumenthal and Getoor [BG68], Ethier and Kurtz [EK85], Renuv and Yor [RY99], and others. On the other hand, development of a complete theory of quantum stochastics such as quantum Markov processes has been progressing at a much slower pace in comparison with that of its classical counterpart. This is perhaps because development of a theory of quantum stochastics and quantum Markov processes requires an unusually large number of tools from operator theory and perhaps also because the probabilistic and analytical tools for understanding sample path behaviors of quantum stochastic processes have yet to be developed.

The main goal of this monograph is to give a systematic exploration of relevant topics in quantum Markov processes in hopes of stimulating further research along this line and of stimulating interest in the classical stochastics community for research in its quantum counterpart. This monograph is written largely based on the current account of relevant research results by widely surveying relevant results contributed by many prominent researchers in quantum probability, quantum noise, quantum stochastic calculus, stochastic quantum differential equations, quantum Markov semigroups, strong quantum Markov processes, and large time asymptotic behaviors of quantum Markov semigroups through a systematic and self-contained introduction/presentation of these very interesting topics in an attempt to illuminate the rigor and beauty of quantum stochastics and quantum Markov processes.

The intended readers for this research monograph can include but are not limited to the following 3 groups of researchers: (i) classical probabilists and stochastic analysts who are interested in learning and extending their research to quantum probability and quantum stochastic processes; (ii) operator theorists who are interested in linear operators acting on C^*-and von Neumann algebras and their applications to quantum systems; and (iii) statistical, theoretical, and quantum physicists who are interested in a rigorous presentation of a mathematical theory of quantum stochastics and quantum Markov processes.

This monograph can be used as an introduction or a research reference for advanced graduate students and researchers who have been exposed to theory of classical (or commutative)

Markov processes and who also have special interest in its noncommutative counterpart. With a few exceptions, this monograph is intended to be as much self-contained as possible by providing necessary review material and the proofs for almost all of the lemmas, propositions, and theorems contained herein. Some knowledge in real analysis, functional analysis, and classical stochastic processes will be helpful. However, no background material is assumed beyond knowledge of the basic theory of Hilbert spaces, bounded linear operators, and classical Markov processes.

This monograph consists of 11 chapters that constitute the backbone of quantum stochastics and quantum Markov processes. The content of each of these 11 chapters are briefly summarized below.

Chapter 1. Operator Algebras and Topologies

In preparing the tools that are required for developing theory of quantum stochastics and quantum Markov processes, this chapter gives a brief review of complex Hilbert spaces and their topological dual spaces together with the concepts of weak and strong convergence. The concepts of bounded and unbounded linear operators on complex Hilbert spaces are introduced. Special classes of bounded linear operators including self-adjoint, trace-class, compact and projection operators, operator-valued spectral measures, and the von Neumann spectral representation are discussed. Various concepts of operator topologies, such as norm-topology, strong topology, weak topology, σ-strong topology, σ-weak topology, and $weak^*$-topology on the space of bounded linear operators are given. It is illustrated that some of these topologies are actually equivalent under appropriate conditions. This chapter also introduces 2 major types of algebras, namely, the C^*-algebra and von Neumann algebra of operators on a complex Hilbert space. These 2 different types of algebras are all to be denoted by $\mathcal{A}$. It is assumed throughout the monograph that all algebras are unital, i.e., they contain the identity operator on the Hilbert space. These algebras, especially the von Neumann algebra, are important tools for describing a quantum system. One of the important topology on $\mathcal{A}$ that plays an important role in studying quantum Markov semigroups or quantum Markov processes is the so-called σ-weak continuity. Finally, representation of a C^*-algebra is defined and the background material for describing Gelfand-Naimark-Segal construction for a representation of C^*-algebras is described in details.

Chapter 2. Quantum Probability

Complex Hilbert spaces play an important role in describing quantum systems. In fact, with every quantum system there is a corresponding complex Hilbert space $\mathbb{H}$ that consists of the states of the quantum system. The Hilbert space $\mathbb{H}$ that represents a composite quantum system of n subsystems can be expressed as the tensor product $\mathbb{H}_1 \otimes \ldots \otimes \mathbb{H}_n$ of n component systems described, respectively, by $\mathbb{H}_1, \ldots, \mathbb{H}_{n-1}$, and $\mathbb{H}_n$. This chapter provides mathematical formulation of a generic quantum system according to the following set of postulates initiated by von Neumann (see, e.g., von Neumann [vNeu55]):

Postulate 1 With every *quantum system* there is a corresponding finite-dimensional or infinite-dimensional separable complex Hilbert space $\mathbb{H}$ on which a C^*- or a von Neumann algebra of linear operators $\mathcal{A}$ is defined. This complex Hilbert space $\mathbb{H}$ is called in physics terminology the *space of states*.

Postulate 2 Given a C^*- or a von Neumann algebra of operators $\mathcal{A}$ on $\mathbb{H}$ for the quantum system, the space of quantum states $\mathcal{S}(\mathcal{A})$ of the quantum system then consists of all positive (and hence self-adjoint) trace class operators $\rho \in \mathcal{A}$ with unit trace, $tr(\rho) = 1$. The *pure states* are projection operators onto one-dimensional subspaces of $\mathbb{H}$, and the *mixed states* are those that can be written as convex combination of pure states. A state ρ will be called the *density operator* or *density matrix* if $tr(\rho\mathbf{a}) = \mathrm{tr}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{A}$. Density operators play an important role in quantum physics. The pair $(\mathcal{A}, \rho)$ of a von Neumann algebra $\mathcal{A}$ and a quantum state ρ will be called a quantum probability space. Roughly speaking, $\mathcal{A}$ represents quantum random variables or obervables, and ρ represents the probability law that governs the quantum system.

Postulate 3 Roughly speaking, an observable of the quantum system is a positive operator-valued measure $\mathbf{a}$ defined on the Borel measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Specifically, for each Borel set $B \in \mathcal{B}(\mathbb{R})$, $\mathbf{a}(B)$ is a self-adjoint linear (but not necessarily bounded) operator on the Hilbert space $\mathbb{H}$.

Postulate 4 A process of measurement in a quantum system is the correspondence between the observable-state pair $(\mathbf{a}, \rho)$ and the probability measure $\mu_{\mathbf{a}}$ on the real Borel measurable space associated with the observable $\mathbf{a}$. For every Borel subset $E \in \mathcal{B}(\mathbb{R})$, the quantity $0 \le \mu_{\mathbf{a}}(E) \le 1$ is the probability that when a quantum system is in the state ρ, the result of the measurement of the observable $\mathbf{a}$ belongs to E. The expectation value (the mean value) of the observable $\mathbf{a}$ in the state ρ is

$$\langle \mathbf{a} | \rho \rangle = \int_{-\infty}^{\infty} \lambda d\mu_{\mathbf{a}}(\lambda),$$

where $\mu_{\mathbf{a}}(\lambda) = \mu_{\mathbf{a}}((-\infty, \lambda))$ is the distribution function for the probability measure $\mu_{\mathbf{a}}$.

The mathematical details of each of the above 4 postulates constitute the topics discussed in this chapter. In particular, the concept of quantum probability space, quantum random variable, quantum expectations, and quantum conditional expectations are introduced. It is well known that the concept of an expectation operator conditioned on a given sub-σ-algebra plays a crucial role in classical Markovian theory of processes. Similar to the classical Markovian properties, the concept of quantum Markovian properties is hinged heavily on the concept of conditional expectation of an observable with respect to a von Neumann sub-algebra. Unfortunately, a complete theory of quantum conditional expectation is yet to be developed. While there is very little literature published in this area, this chapter gives, in addition to weak conditional expectation, the definition and a construction of quantum conditional expectation given a von Neumann sub-algebra. These concepts are sufficient for us to develop quantum Markovian properties in the subsequent chapters.

Chapter 3. Quantum Stochastic Calculus

This chapter begins with introductions of symmetric Fock space $\Gamma(\mathbb{H})$ and symmetric Guichardet space $\Phi(\mathbb{H})$ of a generic complex Hilbert space $\mathbb{H}$. It is shown that these 2 spaces are actually isomorphic and have been used interchangeably throughout the chapter. In particular, the simple and yet useful integral-summation formula for the Guichardet space is often employed to establish the essential results in quantum stochastic calculus. It has been shown that the symmetric Fock space (and hence the symmetric Guichadet space) provides a plausible mathematical tool for modeling phenomena in quantum optics or quantum electrodynamics. Many results in the quantum physics and quantum probability literature are actually established based on this concrete model space. When the generic complex Hilbert space $\mathbb{H} = L^2(\mathbb{R}_+; \mathbb{K})$ (where $\mathbb{K}$ is another complex Hilbert space), the class of exponential vectors along with 3 different types of quantum noise processes, namely, the creation, annihilation, and neutral processes, can be introduced. It is shown that the subspace generated by the class of exponential vectors is dense in the Fock space. Therefore, it is convenient to verify the properties that hold for a Fock space by verifying the same hold for the class of exponential vectors. Parallel to those of Itô integrals with respect to classical Brownian motion and/or Poisson process, the concepts of a quantum stochastic integral of a operator-valued process (as a member of the Fock space) with respect to each of the above mentioned quantum noise processes are constructed. The quantum stochastic calculus, parallel to those of classical Itô calculus, is developed within the content of Fock or Guichadet space. The quantum stochastic calculus enables more detailed analysis of quantum stochastic differential equations, which is the main topic of discussion in Chapter 4.

Chapter 4. Quantum Stochastic Differential Equations
Based on the results in Chapter 3, this chapter derives and considers a general form of linear (left) and (right) Hudson-Parthasarathy quantum stochastic differential equations driven by quantum noises in symmetric Fock space and with operator-valued matrices as coefficients. Specifically, this chapter studies the existence and uniqueness of the solution process for both the left and right quantum stochastic differential equations and conditions under which the solution processes are unitary, contraction, isometry, and co-isometry. These results make extensive use of the properties of stochastic integral driven by quantum noise and its quantum stochastic calculus in the context of a symmetric Fock space. In this chapter various discrete approximation schemes of the left Hudson-Parthasarathy QSDE are explored for numerical computation. Specifically, it is shown that the solution of the Hudson-Parthasarathy QSDE can be approximated by a sequence of discrete interaction models with decreasing time step. In order to study this problem, discrete interaction models are embbeded in a limiting space. This allows us to prove strong convergence of the embedded discrete cocycles to the solution of the Hudson-Parthasarathy QSDE. It is also pointed out that the way in which the embedding is done does not affect the proof the main results presented in this chapter.

Chapter 5. Quantum Markov Semigroups
This chapter defines and explores basic properties of a quantum Markov semigroup $\{\mathfrak{T}_t, t \geq 0\}$ of linear maps on the C^*-algebra or von Neumann algebra $\mathcal{A}$. The quantum Markov semigroup (QMS) plays a key role in describing quantum Markov processes, which are to be explored in the subsequent chapters. The concept of QMS extends the semigroup of probability transition operators $\{T_t, t \geq 0\}$ for a classical Markov process. In the case that the QMS $\{\mathfrak{T}_t, t \geq 0\}$ is uniformly continuous and $\mathcal{A} = \mathcal{L}^\infty(\mathbb{H})$ (the space of bounded linear operators on $\mathbb{H}$), then its infinitesimal generator $\mathfrak{L}\colon \mathcal{D}(\mathfrak{L}) \to \mathcal{A}$ can be completely characterized by the celebrated Lindblat theorem. In this case, the evolution of quantum states $\{\rho_t, t \geq 0\}$ can then be described by the Lindblat master equation $\dot{\rho}_t = \mathfrak{L}(\rho_t)$ based on which many advances in quantum systems have been made.

Chapter 6. Minimal QDS

The previous chapter dealt with general QMSs and with the characteristics of the infinitesimal generator $\mathfrak{L}$ for a given quantum Markov semigroup (QMS) $\{\mathfrak{T}_t, t \geq 0\}$ of operators that are uniformly continuous. However, the class of uniformly continuous quantum Markov semigroups is too small for applications in quantum probability and quantum physics. Construction of the quantum Markov semigroup (QMS) based on 2 infinitesimal generators **G** and **L** that appear in the Lindbladd master equation is given in this chapter. The problem of constructing quantum Markov semigroups with unbounded generator, in principle, could be treated with the Hille-Yosida theorem (see Yosida [Yos80]) at least in the case when the domain of the infinitesimal generator is an algebra so that conditional complete positivity makes sense. However, in all the applications the infinitesimal generator $\mathfrak{L}$ is not given explicitly but is given formally in a "generalized" Lindblad form with unbounded operators **G** and **L**. We follow in this chapter Davies's construction of the predual semigroup of a quantum Markov semigroup on the von Neumann algebra $\mathcal{L}^\infty(\mathbb{H})$ from given operators **G** and **L** in $\mathbb{H}$.

Chapter 7. Quantum Markov Processes

It is widely known that a classical Markov semigroup of transition operators can be generated from a given classical Markov process. On the other hand, given a Markov semigroup $\{T_t, t \geq 0\}$ a classical Markov process $\{X_t, t \geq 0\}$ can be constructed using the Kolmogorov consistency theorem. In the context of quantum probability, the construction of a quantum Markov process from a given quantum Markov semigroup of operators $\{\mathfrak{T}_t, t \geq 0\}$) turns out to be a nontrivial matter. This is partly due to the fact that, although the general concept of conditional expectation of an observable given a sub-von Neumann algebras $\mathcal{B} \subset \mathcal{A}$ can be defined and required properties can be described, an explicit construction of such a conditional expectation is still unavailable in general. The main objective of this chapter is to introduce relevant concepts and to develop some properties of quantum Markov processes in the content of quantum probability explored in previous chapters. There are 2 major components in this chapter, namely, (i) introduction of concepts and derivation of properties of a quantum Markov processes based on some assumed and/or derived properties of conditional expectation $\mathbf{E}_\rho[\cdot\, |\mathcal{A}_{t]}]$ based on a filtration of sub-von Neumann algebras $\{\mathcal{A}_{t]}, t \geq 0\}$ of $\mathcal{A}$, and (ii) Markov dilation or construction of a weak quantum Markov flow (WQMF) from a given quantum Markov semigroup $\{\mathfrak{T}_t, t \geq 0\}$ using the weak conditional expectation $\mathbf{E}[\cdot\, |\mathbf{F}_t]$, where $\{\mathbf{F}_t, t \geq 0\}$ is a filtration of orthogonal projection operators defined on the complex Hilbert space.

Chapter 8. Strong Quantum Markov Processes

It is well known that the concept of stopping times plays an important role in the classical strong Markov processes. Similarly, our definition of strong quantum Markov processes generalizing the classical strong Markov processes requires introduction of a quantum version of stopping times. A quantum (noncommutative) stopping time (to be abbreviated as QST whenever and wherever is convenient in the following) on a filtered Hilbert space, on the other hand, is defined as a (right continuous) spectral measure on $[0, \infty]$ with values in the space of orthogonal projection operators on $\mathbb{H}$ that satisfy some appropriate adaptivity properties. In the above, the weak filtration of projection operators $\{\mathbf{F}_t, t \geq 0\}$ plays the role of filtration of sub-σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ in the classical stopping times and classical strong

Markov processes. For a classical stopping time τ, let $\mathcal{F}_\tau$ be the collection of events anterior to τ. Similar to $\mathcal{F}_\tau$, the corresponding projection operator $\mathbf{F}_\tau$ for the quantum version is then first defined for a discrete QST τ and then for the general QST τ via convergence of $\mathbf{F}_{\tau_n}$ by a sequence of discrete QSTs $(\tau_n)_{n=1}^\infty$. This chapter provides the concept of quantum Markov flows $(\mathbb{H}, \mathbf{F}_t, j_t, t \geq 0)$ introduced in Chapter 7 to strong quantum Markov flow. Examples of strong quantum Markov flows on the symmetric Fock space $\Gamma = \Gamma_{sym}(L^2(\mathbb{R}_+; \mathbb{K}))$ are also given. Sufficient conditions for strong quantum Markovian flows are established. If the quantum Markov semigroup $\{\mathfrak{T}_t, t \geq 0\}$ is uniformly continuous (and hence its infinitesimal generator $\mathfrak{L}$ exists), then the process $\{\mathfrak{M}_t(\mathbf{X}), t \geq 0\}$, $\mathbf{X} \in \mathcal{D}(\mathfrak{L})$, is a quantum martingale in the sense that $\mathbf{F}_s \mathfrak{M}_t(\mathbf{X}) \mathbf{F}_s = \mathfrak{M}_s(\mathbf{X})$ for all $s \geq t$, where $\mathfrak{M}_t(\mathbf{X})$ is defined by

$$\mathfrak{M}_t(\mathbf{X}) = j_t(\mathbf{X}) - j_0(\mathbf{X}) - \int_0^t j_u(\mathfrak{L}(\mathbf{X}))du.$$

Additionally, a noncommutative generalization of a well-known Dynkin's formula is explored. In this chapter, we develop the theory of quantum stopping times, quantum martingales, and their corresponding properties that are parallel to the classical theory briefly described above. Lyapunov stability of strong quantum Markov processes based on the symmetric Fock space is also discussed.

Chapter 9. Invariant Normal States

The main purpose of this chapter is to develop the concept of invariant normal states via the fixed points of the QMS $\{\mathfrak{T}_t, t \geq 0\}$ and $\{\mathfrak{T}_{*t}, t \geq 0\}$. The invariant normal states turns out to be an extension of stationary measure of a classical Markov process. In this chapter we first examine the extension of the classical Prohorov theorem to its quantum counterpart and then explore the existence conditions for an invariant normal state under the general QMS and then under the uniformly continuous QMS. It is proved that if the QMS $\{\mathfrak{T}_t, t \geq 0\}$ possesses a faithful invariant normal state ρ, then ρ is unique. In addition, properties of von Neumann sub-algebras $\mathcal{F}(\mathfrak{T})$ and $\mathcal{N}(\mathfrak{T})$ of $\mathcal{A}$ that play important roles in the next 2 chapters are also examined.

Chapter 10. Recurrence and Transience

This chapter explores concepts and surveys current results on the recurrence and transience of quantum Markov semigroups obtained by a few major contributors, including F. Fagnola, Robelledo, Umaneta, and others. Transience and recurrence come to a probabilist mind as the first step in the classification of Markov processes. In classical probability, recurrence and transience have been extensively studied in connection with semigroup and potential theory. In this chapter, the connection between the potential theory and recurrence and transience has been extended to quantum Markov semigroups. Specifically, a potential associated to the QMS $\{\mathfrak{T}_t, t \geq 0\}$ defined by $\mathfrak{U}: \mathcal{A} \to \mathcal{A}$ as $\mathfrak{U}(\mathbf{a}) = \int_0^\infty \mathfrak{T}_t(\mathbf{a})dt$ for $\mathbf{a} \in \mathcal{A}$ is explored. According to the nature of $\mathfrak{U}(\mathbf{a})$, the properties of transience or recurrence for the QMS $\{\mathfrak{T}_t, t \geq 0\}$ are characterized. In addition, positive recurrent projections is defined via support projections of stationary normal states. Then we explore its main related properties as, for instance, the relation with sub- (or super-) harmonic operators and the dichotomy transience recurrence for irreducible semigroups. It is shown that an irreducible quantum Markov semigroup is either recurrent or transient and characterizes transient semigroups by means of the existence of nontrivial superhaimonic operators. This chapter also explores its main related properties such as, for instance, the relation with sub- (or super-) harmonic operators and the dichotomy transience recurrence for irreducible semigroups.

Chapter 11. Ergodic Theory

This chapter develops the ergodicity, mean ergodicity, and statistical stability for the QMS $\{\mathfrak{T}_t, t \geq 0\}$ on the von Neumann algebra $\mathcal{A}$ and its associated semigroup $\{\mathfrak{T}_{*t}, t \geq 0\}$ on the predual $\mathcal{A}_*$. The QMS $\{\mathfrak{T}_{*t}, t \geq 0\}$ is said to be ergodic if w-$\lim_{t\to\infty} \mathfrak{T}_{*t}(\omega)$ exists for every $\omega \in \mathcal{A}_*$ and mean ergodic if w-$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathfrak{T}_{*s}(\omega)ds$ exists for every $\omega \in \mathcal{A}_*$. It is clear that ergodicity implies mean ergodicity and that there is a close connection between the ergodicity of $\{\mathfrak{T}_{*t}, t \geq 0\}$ with its invariant normal states explored in Chapter 11. In fact, it is shown in this chapter that every w-limit of the Cesaro means $\left(\frac{1}{t}\int_0^t \mathfrak{T}_{*s}(\omega)ds\right)$ is an invariant state for any quantum stateω. Conversely, if there exists a faithful family $\mathcal{G}$ of normal states then the QMS $\{\mathfrak{T}_t, t \geq 0\}$ (and hence $\{\mathfrak{T}_{*t}, t \geq 0\}$) is mean ergodic. In addition, equivalent conditions for mean ergodicity of the QMS are established in this chapter. While these equivalent conditions serve as a characterization of mean ergodicity, they are very difficult to verify especially in the infinite dimensional case. To overcome this shortcoming, Emel'syanov and Wolff [EW06] introduced the quantum version of a mean lower bound for a positive quantum state and proved that if the distance between a Cesaro mean of any normal state can be made asymptotically closed to the order interval of a mean lower bound element, then the QMS $\{\mathfrak{T}_{*t}, t \geq 0\}$ is mean ergodic. Furthermore, if $\mathcal{A}$ is atomic, then the space of fixed points $\mathcal{F}(\mathfrak{T}_*)$ is finite dimensional. On the other hand it is also shown that the QMS is mean ergodic and the space of invariant states is one dimensional if and only if there exists a nontrivial mean lower bound. A new proof using GNS representation of states for Frigerio-Verri's theorem (see Frigerio and Verri [FV82]) that addressed the sufficient conditions for the ergodicity of the QMS $\{\mathfrak{T}_t, t \geq 0\}$ is also provided in this chapter. Finally, this chapter closes with a result that connects the existence of a mean lower bound with statistical stability of the QMS.

While it is recommended that the chapters be read in succession for readers who are exposed to the subject for the first time, we outline the flow of presentations of the chapters below for the benefit of readers who are interested only in certain topic areas for a quick overview/reading.

- Follow Chapter 1 ⇒ Chapter 2 ⇒ Chapter 3 ⇒ Chapter 4, for exploration of quantum stochastic calculus and quantum stochastic differential equations.
- Follow Chapter 1 ⇒ Chapter 2 ⇒ Chapter 3 ⇒ Chapter 5 ⇒ Chapter 6 ⇒ Chapter 7 ⇒ Chapter 8 ⇒ Chapter 9 ⇒ Chapter 10 ⇒ Chapter 11 for quantum Markov semigroups/processes and their large time asymptotic behaviors.
- Follow Chapter 1 ⇒ Chapter 2 ⇒ Chapter 3 for introduction to quantum probability.

1

Operator Algebras and Topologies

This chapter serves as an overview of some of the basic building blocks for quantum probability, quantum Markov semigroups/processes, and their large time asymptotic behavior that are to follow.

We start out with a brief review of complex Hilbert spaces and their topological dual spaces together with the concepts of weak and strong convergence. The concepts of linear operators on complex Hilbert spaces are introduced. Special classes of bounded linear operators including self-adjoint, Hilbert-Schmidt, trace-class, compact and projection operators, operator-valued spectral measures, and the celebrated spectral representation theorem due originally to von Neumann (see von Neumann [vNeu55]) are discussed. We also define various concepts of operator topologies, such as norm-topology, strong topology, weak topology, σ-strong topology, σ-weak topology, and weak*-topology, on the space of bounded linear operators. Equivalence of some of these topologies under appropriate conditions are illustrated. This chapter also introduces the 2 major types of algebras, namely, the C^*-algebra and von Neumann algebra of operators on a complex Hilbert space. These 2 different types of algebras are all to be denoted by $\mathcal{A}$. However, the results will be stated with specification to which of the algebras is under consideration. Unless otherwise stated, it is assumed throughout the book that all algebras are unital; i.e., they contain the identity operator. These algebras, especially the von Neumann algebra, are important tools for describing quantum probability spaces and quantum systems. Many of the results presented in this chapter are stated in terms of C^*-algebras in general without specifications to the von Neumann algebra. One of the important topology on $\mathcal{A}$ that plays an important role in studying quantum Markov semigroups or quantum Markov processes is the so called σ-weak continuity. Finally, we define representation of a C^*-algebra and prepare the background material for describing Gelfand-Naimark-Segal construction for a representation of C^*-algebra, which is described in detail.

The material presented in this chapter can be found in most of the research monographs or graduate texts on functional analysis, such as those of Rudin [Rud91], Conway [Con94], Reed and Simon [RS70], [RS75], and Yosida [Yos80]. Properties of operator algebras including those of C^*- and von Neumann algebras and their GNS construction/representation can be found in Takesaki [Tak76], Bratteli and Robinson [BR87], and Dixmier [Dix81].

1.1 Complex Hilbert Spaces

This section serves as a review of complex Hilbert spaces. With a few exceptions, widely known theorems and/or propositions are stated without a proof. The material presented in this section can be found in most of standard functional analysis textbooks or monographs.

Throughout the end of this book, let $\imath = \sqrt{-1}$, and let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real numbers and complex numbers, respectively. If $z = x + \imath y \in \mathbb{C}$, where $x, y \in \mathbb{R}$, let $\bar{z} = x - \imath y \in \mathbb{C}$ and $|z| = \sqrt{x^2 + y^2} \in \mathbb{R}_+ := \{a \in \mathbb{R} \mid a \geq 0\}$ denote the complex conjugate and the modulus of the complex number $z \in \mathbb{C}$, respectively. In this case, $x = \Re(z)$ is the real part of z and $y = \Im(z)$ is the imaginary part of z. Elements in $\mathbb{R}$ or $\mathbb{C}$ shall be denoted by lower case letters such a, b, or x and y, etc.

For $-\infty < a < b < \infty$, we use the usual convention for closed, open, and half-open *intervals* on the real line $\mathbb{R}$ such as $[a, b]$, $[a, b[$, $]a, b]$, $]a, b[$, $]-\infty, a]$, $]-\infty, a[$, $[b, \infty[$, and $]b, \infty[$ throughout this book.

Let $\mathbb{H}$ denote a (generic) Hilbert space over the field of complex numbers $\mathbb{C}$ and be referred to as a complex Hilbert space. The complex Hilbert space $\mathbb{H}$ will be equipped with the Hilbertian inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ that satisfies the following conditions:

1. (Linearity in second argument)
$$\langle \phi, a\varphi + b\varsigma \rangle_{\mathbb{H}} = a\langle \phi, \varphi \rangle_{\mathbb{H}} + b\langle \phi, \varsigma \rangle_{\mathbb{H}}, \quad \forall a, b \in \mathbb{C} \text{ and } \forall \phi, \varphi, \varsigma \in \mathbb{H}$$
2. (Conjugate-linearity in the first argument)
$$\langle a\phi, \varphi \rangle_{\mathbb{H}} = \bar{a}\langle \varphi, \phi \rangle_{\mathbb{H}}, \quad \forall a \in \mathbb{C}, \quad \forall \phi, \varphi \in \mathbb{H}$$
3. (Conjugate symmetric)
$$\langle \phi, \varphi \rangle = \overline{\langle \varphi, \phi \rangle}, \quad \forall \phi, \varphi \in \mathbb{H}$$
4. (Positive definiteness)
$$\langle \phi, \phi \rangle_{\mathbb{H}} \geq 0, \quad \forall \phi \in \mathbb{H}, \text{ and } \langle \phi, \phi \rangle = 0 \text{ if and only if } \phi = 0.$$

We comment here that conventions for Hilbertian inner product differ as to which argument should be linear and which should be conjugate-linear. Throughout this book, we take the first to be conjugate-linear and the second to be linear. This is the convention used by essentially all physicists and originates in Dirac's bra-ket notation (see Chapter 2) in quantum mechanics. The opposite convention is more common in mathematics.

The Hilbertian norm $\| \cdot \|_{\mathbb{H}} : \mathbb{H} \to \mathbb{R}$ corresponding to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ is defined by

$$\|\psi\|_{\mathbb{H}} = \sqrt{\langle \psi, \psi \rangle_{\mathbb{H}}}, \quad \forall \psi \in \mathbb{H}.$$

It is known that a complex Hilbert space is a complex Banach space under the Hilbertian norm. However, we occasionally will also work with a complex Banach space $\mathbb{X}$ equipped with the norm $\| \cdot \|_{\mathbb{X}} : \mathbb{X} \to \mathbb{R}$ which may not be equipped with a Hilbertian inner product (and therefore is not a Hilbert space). Instead they may be equipped with a semi-inner product. As a reference we recall that a semi-inner product for a complex Banach space $\mathbb{X}$ is a function $[\cdot, \cdot] : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ such that

1. $[\phi + \varphi, \psi] = [\phi, \psi] + [\varphi, \psi], \quad \forall \phi, \varphi, \psi \in \mathbb{X}$
2. $[\phi, a\varphi] = a[\phi, \varphi], \quad \forall a \in \mathbb{C}$ and $\forall \phi, \varphi \in \mathbb{X}$
3. $[a\phi, \varphi] = \bar{a}[\phi, \varphi], \quad \forall a \in \mathbb{C}$ and $\forall \phi, \varphi \in \mathbb{X}$
4. $[\phi, \varphi] = \overline{[\varphi, \phi]}, \quad \forall \phi, \varphi \in \mathbb{X}$

5. $[\phi, \phi] \geq 0 \quad \forall \phi \in \mathbb{X}$ and $[\phi, \phi] = 0$ if $\phi = 0$
6. $|[\phi, \varphi]| \leq [\phi, \phi]^{1/2}[\varphi, \varphi]^{1/2}, \quad \forall \phi, \varphi \in \mathbb{X}$.

It is clear that a Hilbertian inner product is a semi-inner product. However, a semi-inner product is not necessary a Hilbertian inner product. The difference is that a semi-inner product satisfies all the properties of inner products except that it is not required to be strictly positive. The seminorm associated with the semi-inner product is a function $\|\cdot\|: \mathbb{X} \to \mathbb{R}$ is defined by

$$\|\phi\|^2 = [\phi, \phi], \quad \forall \phi \in \mathbb{X}.$$

Therefore, $\|\phi\| = 0$ does not necessarily imply that $\phi = 0$ for a semi-norm.

Throughout the end, elements (or vectors) of a complex Hilbert space $\mathbb{H}$ or a complex Banach space $\mathbb{X}$ will be denoted by lower case Greek symbols such as ϕ, φ, and ζ, and, occasionally, the lower case letters such as u, v, and w. The zero vector of $\mathbb{H}$ shall be denoted by 0.

The closure of a subset A of $\mathbb{H}$ (or $\mathbb{X}$) in the Hilbertian (or Banach) norm $\|\cdot\|$ is denoted by $\bar{A}$. A subset A of $\mathbb{H}$ (or $\mathbb{X}$) is said to be dense in $\mathbb{H}$ (or in $\mathbb{X}$) if $\bar{A} = \mathbb{H}$ (or $\bar{A} = \mathbb{X}$). The complex Hilbert space $\mathbb{H}$ (or the Banach space $\mathbb{X}$) is said to be separable if it contains a countable dense subset. A subset A of a Hilbert space $\mathbb{H}$ (or Banach space $\mathbb{X}$) is said to be total in $\mathbb{H}$ (or in $\mathbb{X}$) if $[A]$, the linear manifold generated by A, is dense in $\mathbb{H}$ (or in $\mathbb{X}$).

Two vectors ψ and ϕ in complex Hilbert space $\mathbb{H}$ are called *orthogonal* if $\langle \psi, \phi \rangle = 0$. In this case, we denote $\psi \perp \phi$. A set $A \subset \mathbb{H}$ is called an orthogonal set of vectors if $\psi \perp \phi$ for all $\psi, \phi \in A$ and $\psi \neq \phi$. An orthogonal set $A \subset \mathbb{H}$ is an orthonormal set if $\|\psi\| = 1$ for all $\psi \in A$. An *orthonormal basis* A for $\mathbb{H}$ is a maximal orthonormal set, i.e., if $B \subset \mathbb{H}$ is such that $A \subset B$ then B is not an orthonormal set. The Hilbert space $\mathbb{H}$ is said to be N dimensional if the orthonormal basis A consists of N distinct elements (vectors). The Hilbert space $\mathbb{H}$ is said to be infinite dimensional if its orthonoral basis consists of infinite distinct vectors.

Naturally, we do not speak of the orthogonality in a non-Hilbertian Banach space $\mathbb{X}$ because it is not equipped with an inner product.

Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of all *natural numbers*, i.e., positive integers.

Some of the widely known and frequently used Hilbert spaces are given below.

Example 1 $\mathbb{C}^N$, the space of N-component complex vectors, is an N-dimensional Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle: \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ defined by $\langle a, b \rangle = \sum_{i=1}^{N} \overline{a_i} b_i$ for all $a = (a_1, a_2, \ldots, a_N)$ and $b = (b_1, b_2, \ldots, b_N)$ in $\mathbb{C}^N$, where $\overline{a_i}$ is the complex conjugate of a_i.

Example 2 The space of squared summmable complex sequences,

$$l^2(\mathbb{N}; \mathbb{C}) = \left\{ (a_n)_{n \geq 1} \Big| \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\},$$

is an infinite-dimensional complex Hilbert space equipped with the inner product

$$\langle \cdot, \cdot \rangle: l^2(\mathbb{N}; \mathbb{C}) \times l^2(\mathbb{N}; \mathbb{C}) \to \mathbb{C}$$

defined by $\langle (a_n)_{n \geq 1}, (b_n)_{n \geq 1} \rangle = \sum_{n=1}^{\infty} \overline{a_n} b_n$ for all $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \in l^2(\mathbb{N}; \mathbb{C})$. From functional analysis point of view, all infinite-dimensional complex Hilbert spaces are equivalent