Commutative Algebra and Noncommutative Algebraic Geometry, I
MSRI Publications
Volume **67**, 2015

Growth functions

JASON P. BELL

We give a survey of the use of growth functions in algebra. In particular, we define Gelfand–Kirillov dimension and give an overview of some of the main results about this dimension, including Bergman's gap theorem, the solution of the Artin–Stafford conjecture by Smoktunowicz, and the characterization of groups of polynomially bounded growth by Gromov. In addition, we give a summary of the main ideas employed in the proof of Gromov's theorem and discuss the work of Lenagan and Smoktunowicz, which gives a counterexample to Kurosh's conjecture with polynomially bounded growth.

1. Introduction

The notion of growth is a fundamental object of study in the theory of groups and algebras, due to its utility in answering many basic questions in these fields. The concept of growth was introduced by Gelfand and Kirillov [1966] for algebras and by Milnor [1968] for groups, who showed that there is a strong relation between the growth of the fundamental group of a Riemannian manifold and its curvature. After the seminal works of Gelfand and Kirillov and of Milnor, the study of growth continued and many important advances were made. In particular, Borho and Kraft [1976] further developed the theory of growth in algebras, giving a systematic study of the theory of Gelfand–Kirillov dimension. In addition to this, Milnor [1968] and Wolf [1968] gave a complete characterization of solvable groups with polynomially bounded growth (see Section 2 for relevant definitions).

The reason for the importance of Gelfand–Kirillov dimension, Gelfand–Kirillov transcendence degree, and corresponding notions in the theory of groups is that it serves as a natural noncommutative analogue of Krull dimension (resp. transcendence degree) and thus provides a suitable notion of dimension for noncommutative algebras. Indeed, the first application of Gelfand–Kirillov dimension was to show that the quotient division algebras of the m-th and n-th Weyl algebras are isomorphic if and only if $m = n$, by showing that their transcendence degrees differed when $n \neq m$. Since this initial application,

The author was supported by NSERC grant 31-611456.

the theory of growth has expanded considerably and this notion now plays a fundamental role in both geometric group theory and noncommutative projective geometry, where it serves as a natural notion of dimension.

The objective of these notes is to give a survey of the foundational results of Gelfand–Kirillov dimension as well as related growth functions in the theory of groups and rings which have been used to answer difficult questions. In Section 2, we give an overview of the basic terminology that we will be using throughout. In Section 3, we define Gelfand–Kirillov dimension and give a survey of the most important results in the theory of growth, and in Section 4 we give some of the important results in the theory of combinatorics on words and their application to growth of algebras; in particular, we prove Bergman's gap theorem, which asserts that no algebras whose growth is subquadratic but faster than linear can exist.

In Section 5, we show that Gelfand–Kirillov dimension is a noncommutative analogue of Krull dimension for finitely generated algebras and discuss algebras of low Gelfand–Kirillov dimension. In particular, we discuss the Small–Stafford–Warfield theorem [Small et al. 1985] saying that finitely generated algebras of Gelfand–Kirillov dimension satisfy a polynomial identity and we give a brief discussion of what is known about algebras of Gelfand–Kirillov dimension two. In Section 6, we discuss the ingredients in the proof of Gromov's theorem, which beautifully characterizes the finitely generated groups whose group algebras have finite Gelfand–Kirillov dimension; namely, Gromov's theorem asserts that such groups must have a finite-index subgroup that is nilpotent. In Section 7 and Section 8, we discuss two relatively recent advances in the study of growth; in Section 7, we discuss constructions, mostly developed by Smoktunowicz, which show how to construct pathological examples of algebras of finite Gelfand–Kirillov dimension; in Section 8, we discuss Zhang's so-called lower transcendence degree and its applications to the study of division algebras. Finally, in Section 9, we give a brief overview of Artin's conjecture on the birational classification of noncommutative surfaces and its relation to growth.

2. Preliminaries

We begin by stating the basic definitions we will be using. We let $\mathscr{C}$ denote the class of maps $f : \mathbb{N} \to \mathbb{N}$ that are monotonically increasing and have the property that there is some positive number C such that $f(n) < C^n$. We say that $f \in \mathscr{C}$ has *polynomially bounded* growth if there is some $d > 0$ such that $f(n) \le n^d$ for all n sufficiently large; we say that f has *exponential growth* if there exists a constant $C > 1$ such that $f(n) > C^n$ for all n sufficiently large. If $f \in \mathscr{C}$ has neither polynomially bounded nor exponential growth then we say it has *intermediate growth*. If $f(n) = \exp(o(n))$ we say that $f(n)$ has *subexponential growth*. Note that according to these definitions, an element of $\mathscr{C}$ can have intermediate growth without having subexponential growth. As an example, let T be the union of sets of the form $\{(2i)!, \ldots, (2i+1)! - 1\}$ as i ranges over the natural numbers. We may define a weakly increasing map $f(n)$ by declaring that $f(0) = 1$ and $f(n+1) = 2f(n)$ if $n \in T$ and $f(n+1) = f(n)$ if $n \notin T$. Then $f((2m)!) \le 2^{(2m-1)!}$ and thus f cannot have exponential growth. On the other hand, $f((2m+1)!) \ge 2^{(2m+1)!-(2m)!}$. Thus when $n = (2m+1)!$ and $m \ge 1$, we have $f(n) \ge (3/2)^n$ and so f does not have subexponential growth according to our definition. (We note that some, perhaps even most, authors take subexponential growth to include growth types such as the one given in this example.)

Given $f \in \mathscr{C}$ of polynomially bounded growth. We define the *degree* of growth to be

$$\deg(f) := \limsup_{n\to\infty} \frac{\log f(n)}{\log n}.$$

We note that if $f(n)$ is asymptotic to Cn^α then this quantity is equal to α, and so this notion coincides with our usual notion of degree in this case.

Given $f, g \in \mathscr{C}$, we say that f is *asymptotically dominated* by g if there natural numbers $k_1, k_2 \ge 1$ such that $f(n) \le k_1 g(k_2 n)$. If f is asymptotically dominated by g and g is asymptotically dominated by f, then we say that the functions are *asymptotically equivalent*. Asymptotically equivalent functions need not be asymptotic to one another in the conventional sense, but polynomial, exponential, intermediate, and subexponential growth are preserved under this notion of asymptotic equivalence. Furthermore, asymptotically equivalent functions of polynomially bounded growth have the same degree of growth. Henceforth, we will only consider functions up to this notion of asymptotic equivalence.

Given a finitely generated group G and a generating set S with the properties that $1 \in S$ and if $s \in S$ then $s^{-1} \in S$, we can construct a *growth function* of G with respect to the generating set S as follows. We let $d_S(n)$ denote the number of distinct elements of G that can be written as a product of n elements of S. Then $d_S(n)$ is an element of $\mathscr{C}$ since $1 \in S$. For example, if $G = \mathbb{Z}^2$ with generators x, y, then if we take $S = \{1, x, x^{-1}, y, y^{-1}\}$ then we have $d_S(n) = \#\{x^i y^j \ : \ |i| + |j| \le n\} = (n+1)^2$. We note that if T is another generating set then since $T^k \supseteq S$ and $S^k \supseteq T$ for some natural number k we have $d_S(n) \le d_T(kn)$ and $d_T(n) \le d_S(kn)$ for all $n \ge 0$. Thus, although two different growth functions need not be equal, they are equal up to asymptotic equivalence. Thus we may speak unambiguously of the growth function of a finitely generated group G.

Similarly, if k is a field and A is a finitely generated k-algebra, we can associate a growth function as follows. Let V be a finite-dimensional subspace of A that generates A as a k-algebra. Then we define

$$d_V(n) := \dim_k\left(\sum_{j=1}^{n} V^j\right).$$

Unless otherwise specified, we assume that our algebras have an identity; in this case we also assume that $1 \in V$ and so we have $d_V(n) = \dim_k(V^n)$. As in the case with groups, if W is another generating subspace then we have that $d_W(n)$ and $d_V(n)$ are asymptotically equivalent and so we again speak unambiguously of the growth function of A. We make the remark that the growth of a group G is equal to the growth of its group algebra $k[G]$ and so it is enough to consider growth of algebras.

3. General results for algebras of polynomially bounded growth

Given a finitely generated k-algebra A of polynomially bounded growth. We recall that we have a degree function associated to its growth. In this setting, the degree function is called the *Gelfand–Kirillov* dimension and is denoted by $\mathrm{GKdim}(A)$. More formally, we have

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log \dim(V^n)}{\log n},$$

where V is a finite-dimensional vector space containing 1 that generates A as a k-algebra.

A related quantity was first used by Gelfand and Kirillov [1966] to show that $D_n \cong D_m$ if and only if $n = m$ where D_n and D_m are respectively the quotient division algebras of the n-th and m-th Weyl algebras. In addition they conjectured that the quotient division algebra of the enveloping algebra of a finite-dimensional algebraic Lie algebra is isomorphic to the quotient division algebra of a Weyl algebra. (This was ultimately shown to be false [Alev et al. 1996].) We note that in the nonfinitely generated case, one simply defines the GK dimension to be the supremum of the GK dimensions of all finitely generated subalgebras.

We now discuss the foundational results in the theory of growth. Before discussing these results in greater detail, we give a quick summary of these results for the reader's convenience. We let k be a field and we let A be a finitely generated k-algebra. Then we have the following:

(i) if $\mathrm{GKdim}(A) \in [0, 1)$ then A is finite-dimensional;

(ii) (Bergman (see [Krause and Lenagan 2000, Theorem 2.5])) if $\mathrm{GKdim}(A) \in [1, 2)$ then A has GK dimension 1;

(iii) (Small, Stafford and Warfield [Small et al. 1985]) if A has GK dimension one then A satisfies a polynomial identity;

(iv) (Small, Stafford, Warfield [Small et al. 1985] and van den Bergh) if A is a domain of GK dimension one and k is algebraically closed then A is commutative;

(v) (Smoktunowicz [2005; 2006]) if A is a graded domain with GK dimension in $[2, 3)$ then A has GK dimension two (and in fact has quadratic growth).

(vi) (Artin and Stafford [1995]) if A is a graded complex domain of GK dimension 2 that is generated in degree one then A is — up to a finite-dimensional vector space — equal to the twisted homogeneous coordinate ring of a curve;

(vii) (Borho and Kraft [1976]) for each $\alpha \in [2, \infty]$ there is an algebra of GK dimension α;

(viii) if A satisfies a polynomial identity and A is semiprime then its GK dimension is equal to a nonnegative integer [Krause and Lenagan 2000, Chapter 10];

(ix) if A is commutative then the GK dimension is equal to the Krull dimension;

(x) (Gromov [1981]) If $A = k[G]$, where G is a finitely generated group, then A has finite GK dimension if and only if G is nilpotent-by-finite;

(xi) (Bass and Guivarch [Krause and Lenagan 2000, Theorem 11.14]) If $A = k[G]$, where G is a finitely generated nilpotent-by-finite group, then the GK dimension of A is an integer given by

$$\sum_i i \cdot d_i,$$

where d_i is the rank of the of the i-th quotient of the lower central series of G.

We note that (i) is immediate since if A is an algebra then we either have $V^i = V^{i+1}$ for some i or we have $V^n \geq n+1$ for all $n \geq 0$. We note that for (x), Gromov [1981] proved that a finitely generated group of polynomially bounded growth is virtually nilpotent and Bass and Guivarch had given the formula for the degree of growth earlier.

4. Combinatorics on words

In this section, we discuss the values that can arise as the GK dimension of an algebra. Many of the foundational results for algebras and groups of low growth come from the theory of combinatorics on words. The reason for this is that given a finitely generated k-algebra A one can associate a monomial algebra

B that has the same growth as A and to determine the growth of a monomial algebra depends on estimating the number of words of length n that avoid a given set of forbidden subwords. We now make this more precise.

A finitely generated algebra A can be written in the form $k\{x_1, \ldots, x_d\}/I$ for some ideal I in the free algebra $k\{x_1, \ldots, x_d\}$. We may put a degree lexicographic ordering on the monomials of the free algebra by declaring that $x_1 \prec x_2 \prec \cdots \prec x_d$. Given an element $f \in k\{x_1, \ldots, x_d\}$, we write f as a linear combination of words in $\{x_1, \ldots, x_d\}$. We then define $\mathrm{in}(f)$, the initial monomial of f, to be the degree lexicographic word w that appears with nonzero coefficient in our expression for f. Then we may associate a monomial ideal J to I by taking the ideal generated by all initial words of elements of I. (Note: it is not sufficient to take the initial words of a generating set for I, as anyone who has worked with Gröbner bases will understand.)

The monomial algebra $B := k\{x_1, \ldots, x_d\}/J$ and A then have identical growth functions, but B has the advantage of having a more concrete way of studying its growth. We record this observation now.

Remark 4.1. Given a finitely generated associative algebra A, there is a finitely generated monomial algebra $B = k\{x_1, \ldots, x_d\}/I$ with identical growth; moreover, if V is the image of the vector space spanned by $\{1, x_1, \ldots, x_d\}$ in B then the dimension of V^n is precisely the number of words over the alphabet $\{x_1, \ldots, x_d\}$ of length at most n that are not in I.

We recall two classical results in the theory of combinatorics of words. These deal with *right infinite words*, which are as one might expect just infinite sequences over some alphabet Σ. The first result is generally called König's infinity lemma, which is very easy to prove, but is nevertheless incredibly useful.

Theorem 4.2 (König). *Let Σ be a finite alphabet and let S be an infinite subset of Σ^*. Then there is a right infinite word w over Σ such that every subword of w is a subword of some word in S.*

The second result is Furstenberg's theorem, which is really part of a more general theorem relating to dynamical systems. We recall that a right infinite word w is *uniformly recurrent* if for any finite subword u that appears in w, there is a natural number $N = N(u)$ with the property that in any block of N consecutive letters in w there must be at least one occurrence of u.

Theorem 4.3 (Furstenberg). *Let Σ be a finite alphabet and let w be a right infinite word over Σ. Then there is a right infinite uniformly recurrent word u over Σ such that every subword of u is also a subword of w.*

The first significant use of the ideas from combinatorics of words in the theory of growth is due to Bergman (see [Krause and Lenagan 2000]), who showed that there is a gap theorem for growth. It is clear that there are no algebras with GK dimension strictly between 0 and 1, but Bergman showed that there are in fact no algebras with GK dimension strictly between 1 and 2. We give a short proof of this theorem.

Theorem 4.4 (Bergman). *There are no algebras of GK dimension strictly between* 1 *and* 2.

Proof. Suppose that A is a finitely generated algebra of GK dimension $\alpha \in (1, 2)$. It is no loss of generality to assume that $A = k\{x_1, \ldots, x_d\}/I$ where I is generated by monomials. Let $\mathcal{S}$ denote the set of words over $\{x_1, \ldots, x_d\}$ that are not in I but have the property that all sufficiently long right extensions are in I. By König's infinity lemma, $\mathcal{S}$ must be a finite set. We let J denote the ideal generated by I and the elements of $\mathcal{S}$. Then $B = k\{x_1, \ldots, x_d\}/J$ and A have the same growth since there is only finite set of words over $\{x_1, \ldots, x_d\}$ that are not in I but are in J. By construction, if W is a word that survives mod J then W has arbitrarily long right extensions that survive mod J.

Let $f(n)$ denote the number of words of length n over $\{x_1, \ldots, x_d\}$ that survive mod J. If V is the span of the images of $1, x_1, \ldots, x_d$ in B then $V^n = 1 + f(1) + \cdots + f(n)$.

There are now two quick cases to consider. If $f(n+1) > f(n)$ for all n then we have $f(n) \geq n+1$ for every natural number n and so $\dim(V^n) \geq \binom{n+2}{2}$ which gives that B has GK dimension at least two, a contradiction.

If $s = f(i+1) = f(i)$ for some i, then we let $W_1, \ldots, W_s$ denote the set of distinct words of length i that are not in J. By our construction of J, each W_j is an initial subword of a word of length $i+1$ that is not in J. Moreover, since there are only s words of length i, we see that each W_j has a unique right extension to a word of length $i+1$ that is not in J. But now we use the fact that each word of length $i+1$ can be written in the form $x_k W_j$ for some $k \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, s\}$. Since W_j has a unique right extension to a word of length $i+1$ that is not in J we see each word of length $i+1$ has a unique extension to a word of length $i+2$ that is not in J. In particular, $f(i+2) = s$. Continuing in this manner, we see that $f(n) = s$ for all $n \geq i$, and so B has linear growth, which is again a contradiction. □

On the other hand, Borho and Kraft [1976] showed that no additional gaps exist in general; that is, any real number that is at least two can be realized as the GK dimension of a finite generated algebra. As an example, we show how one can get an algebra of GK dimension 2.5. We note that this example can be easily modified to get any GK dimension between two and three. Taking polynomial rings over these algebras, one can then construct examples of any GK dimension greater than or equal to two.

We let $A = k\{x, y\}/I$, where I is the ideal generated by all words that have at least three copies of x and all words of the form xy^jx with j not a perfect square. Then the set of words of length at most n over $\{x, y\}$ that are not in I is given by

$$\mathcal{S}_n := \{y^i x y^{j^2} x y^k : i + j^2 + k \le n - 2\} \cup \{y^i x y^j : i + j \le n - 1\} \cup \{y^j : j \le n\}.$$

It is straightforward to check that $\{y^i x y^j : i + j \le n - 1\} \cup \{y^j : j \le n\}$ has size $\binom{n+1}{2} + n + 1$. We note that the set of nonnegative integers for which $i + j^2 + k \le n - 2$ has size at most $2n^{5/2}$ since $i, k \le n - 1$ and $j \le 2\sqrt{n} - 1$. Thus

$$\#\{y^i x y^{j^2} x y^k : i + j^2 + k \le n - 2\} \le 2n^{5/2}.$$

Similarly, since any $i \le (n - 2)/4$, $j \le \sqrt{(n - 2)/4}$, $k \le (n - 2)/4$ satisfies $i + j^2 + k \le n - 2$ we see that

$$\#\{y^i x y^{j^2} x y^k : i + j^2 + k \le n - 2\} \ge (n - 2)^{5/2}/32.$$

Thus

$$\limsup_{n\to\infty} \log(\#\mathcal{S}_n)/\log n = 2.5.$$

It follows that the GK dimension of A is precisely 2.5.

We note that the examples of [Borho and Kraft 1976] are very far from being Noetherian or even Goldie. Smoktunowicz [2005; 2006] showed that if one considers graded domains of GK dimension less than three, then there is a gap.

Theorem 4.5 (Smoktunowicz). *Let A be a finitely generated graded algebra whose GK dimension is in* $[2, 3)$. *Then A has GK dimension* 2.

A partial result of this type had earlier been obtained by Artin and Stafford [1995], who conjectured that the Smoktunowicz gap theorem should hold. Artin and Stafford proved that finitely generated graded algebras whose GK dimension lies in $(2, 11/5)$ could not exist.

5. Small Gelfand–Kirillov dimension

As pointed out earlier, Gelfand–Kirillov dimension can be viewed as a noncommutative analogue of Krull dimension. Much as in the commutative setting special attention has been paid to algebras of small Krull dimension (and, correspondingly, to the study of curves and surfaces and threefolds), there has also been considerable work devoted to the study of algebras of low GK dimension. We first show that GK dimension can be viewed as a reasonable analogue of Krull dimension.

Proposition 5.1. *Let A be a finitely generated commutative k-algebra. Then* $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$.

Proof. Let d denote the Krull dimension of A. By Noether normalization, there exists a subalgebra $B \cong k[x_1, \ldots, x_d]$ of A such that A is a finite B-module. It is straightforward to check that A and B have the same GK dimension. Thus it is enough to prove that $k[x_1, \ldots, x_d]$ has GK dimension d.

Let $C = k[x_1, \ldots, x_d]$ and let $V = k + kx_1 + \cdots + kx_d$. Then V^n has a basis given by all monomials in $x_1, \ldots, x_d$ of total degree at most n. Observe that the monomials $x_1^{i_1} \cdots x_d^{i_d}$ with $i_1 + \cdots + i_d \leq n$ are in one-to-one correspondence with subsets of $\{1, 2, \ldots, n + d\}$ of size d via the rule

$$x_1^{i_1} \cdots x_d^{i_d} \mapsto \{i_1 + 1, i_2 + 2, \ldots, i_d + d\}.$$

Thus V^n has dimension $\binom{n+d}{d}$ which is asymptotic to $n^d/d!$ as $n \to \infty$. Thus the GK dimension of the polynomial ring in d variables is precisely d. The result follows. □

We have seen that algebras of GK dimension 0 are finite-dimensional. While the class of finite-dimensional algebras is not well-understood, the Artin–Wedderburn theorem says that in the prime case all such algebras are given by a matrix ring over a division algebra that is finite-dimensional over its center. In particular, a domain of GK dimension zero over an algebraically closed field is equal to the algebraically closed field. Small, Stafford, and Warfield [1985] proved that a finitely generated algebra of GK dimension one satisfies a polynomial identity. A particularly nice consequence of this, apparently first observed by van den Bergh, shows that a domain of GK dimension one over an algebraically closed field is necessarily commutative.

Theorem 5.2. *Let k be an algebraically closed field and let A be a finitely generated k-algebra that is a domain of GK dimension one. Then A is commutative.*

Proof. Let $t \in A \setminus k$. Then t is not algebraic over k and hence $k[t]$ must be a polynomial ring in one variable. A theorem of Borho and Kraft [1976] shows that A has a quotient division ring D and that D is a finite-dimensional left vector space over $k(t)$. It is straightforward to check that $k(t)$ has GK dimension 1 as a k-algebra and thus D has GK dimension one, since it is a finite module over $k(t)$. We let D act on itself, regarded as a finite-dimensional $k(t)$-vector space, by left multiplication. This gives an embedding of D into $\mathrm{End}_{k(t)}(D)$, which is a matrix ring over $k(t)$. It follows that D satisfies a polynomial identity. Thus D is finite-dimensional over its center. But the center Z of D has the same GK dimension as D since $[D : Z] < \infty$. Hence Z is a field of transcendence degree one. By Tsen's theorem we see that $D = Z$. □

As Stafford and van den Bergh point out, intuitively, this result makes perfect sense: a one-dimensional algebra should be essentially generated by one element and since an element commutes with itself, it is quite reasonable that such algebras should be commutative.

For algebras of Gelfand–Kirillov dimension two, the picture becomes significantly more complicated. For GK dimension two, there is a natural subclass of algebras: algebras of *quadratic growth*. These are finitely generated algebras A of GK dimension two with the property that there is some finite-dimensional generating subspace V of A that contains 1 with the property that the growth of the dimension of V^n is bounded above by Cn^2 for some positive constant C for n sufficiently large.

It is known that once one abandons quadratic growth and considers all algebras of GK dimension two, pathologies arise (see, for example, [Bell 2003]). On the other hand, there are no known examples of prime Noetherian algebras of GK dimension two that do not have quadratic growth. In the case of quadratic growth, algebras appear to be very well-behaved. In [Bell 2010] we showed that a complex domain of quadratic growth is either primitive or it satisfies a polynomial identity. This says that the algebra is either very close to being commutative or, in some sense, as far from being commutative as possible.

6. Gromov's theorem

Gromov's theorem states that every finitely generated group of polynomially bounded growth is nilpotent-by-finite. In this section we will give a brief overview of the ideas used in the proof and discuss possible extensions. We first note that the case of solvable groups of subexponential growth had already been considered by Milnor [1968] and Wolf [1968].

Theorem 6.1 [Milnor 1968; Wolf 1968]. *Let G be a finitely generated solvable group of subexponential growth. Then G is nilpotent-by-finite.*

This result has a completely elementary proof. The first main idea is that a combinatorial argument gives that if G is a finitely generated group of subexponential growth then G' is also finitely generated. From this, one obtains that G is polycyclic. This was Milnor's contribution to the theorem. One now uses induction on the solvable length of G and the fact that conjugation by elements of G on a characteristic finitely generated abelian subgroup gives a linear map. By looking at the eigenvalues of this map, one sees that a dichotomy arises: if one has an eigenvalue whose modulus is strictly greater than 1 then one gets exponential growth; if all eigenvalues have modulus one then Kronecker's theorem gives that they are roots of unity and one can deduce nilpotence of a finite-index subgroup from this. This eigenvalue analysis argument was Wolf's contribution to the argument.