Part I

FUNDAMENTALS OF GEOPHYSICAL FLUID DYNAMICS

To begin at the beginning:
'It is spring, moonless night in the small town,
starless and bible-black, the cobblestreets silent
and the hunched, courters'-and-rabbits' wood limping invisible down
to the sloeblack, slow, black, crowblack, fishingboat-bobbing sea.
Dylan Thomas, *Under Milk Wood*, 1954.

CHAPTER 1

Equations of Motion

HAVING NOTHING BUT A BLANK SLATE, we begin by establishing the governing equations of motion for a fluid, with particular attention to the fluids of Earth's atmosphere and ocean. These equations determine how a fluid flows and evolves when forces are applied to it, or when it is heated or cooled, and so involve both dynamics and thermodynamics. And because the equations of motion are nonlinear the two become intertwined and at times inseparable.

1.1 TIME DERIVATIVES FOR FLUIDS

The equations of motion of fluid mechanics differ from those of rigid-body mechanics because fluids form a continuum, and because fluids flow and deform. Thus, even though the same relatively simple physical laws (Newton's laws and the laws of thermodynamics) govern both solid and fluid media, the expression of these laws differs between the two. To determine the equations of motion for fluids we must clearly establish what the time derivative of some property of a fluid actually means, and that is the subject of this section.

1.1.1 Field and Material Viewpoints

In solid-body mechanics one is normally concerned with the position and momentum of identifiable objects — the angular velocity of a spinning top or the motions of the planets around the Sun are two well-worn examples. The position and velocity of a particular object are then computed as a function of time by formulating equations of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F(\{x_i\}, t), \tag{1.1}$$

where $\{x_i\}$ is the set of positions and velocities of all the interacting objects and the operator F on the right-hand side is formulated using Newton's laws of motion. For example, two massive point objects interacting via their gravitational field obey

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{v}_i, \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = \frac{Gm_j}{(\boldsymbol{r}_i - \boldsymbol{r}_j)^2}\hat{\boldsymbol{r}}_{i,j}, \qquad i = 1, 2;\ j = 3 - i. \tag{1.2}$$

We thereby predict the positions, $\boldsymbol{r}_i$, and velocities, $\boldsymbol{v}_i$, of the objects given their masses, m_i, and the gravitational constant G, and where $\hat{\boldsymbol{r}}_{i,j}$ is a unit vector directed from $\boldsymbol{r}_i$ to $\boldsymbol{r}_j$.

In fluid dynamics such a procedure would lead to an analysis of fluid motions in terms of the positions and momenta of different fluid parcels, each identified by some label, which might simply be their position at an initial time. We call this a *material* point of view, because we are concerned with identifiable pieces of material; it is also sometimes called a *Lagrangian* view, after J.-L. Lagrange. The procedure is perfectly acceptable in principle, and if followed would provide a complete description of the fluid dynamical system. However, from a practical point of view it is much more than we need, and it would be extremely complicated to implement. Instead, for most problems we would like to know what the values of velocity, density and so on are at *fixed points* in space as time passes. (A weather forecast we might care about tells us how warm it will be where we live and, if we are given that, we do not particularly care where a fluid parcel comes from, or where it subsequently goes.) Since the fluid is a continuum, this knowledge is equivalent to knowing how the fields of the dynamical variables evolve in space and time, and this is often known as the *field* or *Eulerian* viewpoint, after L. Euler.[1] Thus, whereas in the material view we consider the time evolution of identifiable fluid elements, in the field view we consider the time evolution of the fluid field from a particular frame of reference. That is, we seek evolution equations of the general form

$$\frac{\partial}{\partial t}\varphi(x, y, z, t) = \mathcal{G}(\varphi, x, y, z, t), \tag{1.3}$$

where the field $\varphi(x, y, z, t)$ represents all the dynamical variables (velocity, density, temperature, etc.) and $\mathcal{G}$ is some operator to be determined from Newton's laws of motion and appropriate thermodynamic laws.

Although the field viewpoint will often turn out to be the most practically useful, the material description is invaluable both in deriving the equations and in the subsequent insight it frequently provides. This is because the important quantities from a fundamental point of view are often those which are associated with a given fluid element: it is these which directly enter Newton's laws of motion and the thermodynamic equations. It is thus important to have a relationship between the rate of change of quantities associated with a given fluid element and the local rate of change of a field. The material or advective derivative provides this relationship.

1.1.2 The Material Derivative of a Fluid Property

A *fluid element* is an infinitesimal, indivisible, piece of fluid — effectively a very small fluid parcel of fixed mass. The *material derivative* is the rate of change of a property (such as temperature or momentum) of a particular fluid element or finite mass of fluid; that is, it is the total time derivative of a property of a piece of fluid. It is also known as the 'substantive derivative' (the derivative associated with a parcel of fluid substance), the 'advective derivative' (because the fluid property is being advected), the 'convective derivative' (convection is a slightly old-fashioned name for advection, still used in some fields), or the 'Lagrangian derivative' (after Lagrange).

Let us suppose that a fluid is characterized by a given velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, which determines its velocity throughout. Let us also suppose that the fluid has another property φ, and let us seek an expression for the rate of change of φ of a fluid element. Since φ is changing in time and in space we use the chain rule,

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z = \frac{\partial\varphi}{\partial t}\delta t + \delta\boldsymbol{x}\cdot\nabla\varphi. \tag{1.4}$$

This is true in general for any δt, δx, etc. The total time derivative is then

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\nabla\varphi. \tag{1.5}$$

If this equation is to represent a material derivative we must identify the time derivative in the second term on the right-hand side with the rate of change of position of a fluid element, namely

its velocity. Hence, the material derivative of the property φ is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\nabla\varphi. \tag{1.6}$$

The right-hand side expresses the material derivative in terms of the local rate of change of φ plus a contribution arising from the spatial variation of φ, experienced only as the fluid parcel moves. Because the material derivative is so common, and to distinguish it from other derivatives, we denote it by the operator $\mathrm{D}/\mathrm{D}t$. Thus, the material derivative of the field φ is

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\partial\varphi}{\partial t} + (\boldsymbol{v}\cdot\nabla)\varphi. \tag{1.7}$$

The brackets in the last term of this equation are helpful in reminding us that $(\boldsymbol{v}\cdot\nabla)$ is an operator acting on φ. The operator $\partial/\partial t + (\boldsymbol{v}\cdot\nabla)$ is the *Eulerian representation of the Lagrangian derivative as applied to a field.* We use the notation $\mathrm{D}/\mathrm{D}t$ rather generally for Lagrangian derivatives, but the operator may take a different form when applied to other objects, such as a fluid volume.

Material derivative of vector field

The material derivative may act on a vector field $\boldsymbol{b}$, in which case

$$\frac{\mathrm{D}\boldsymbol{b}}{\mathrm{D}t} = \frac{\partial\boldsymbol{b}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}. \tag{1.8}$$

In Cartesian coordinates this is

$$\frac{\mathrm{D}\boldsymbol{b}}{\mathrm{D}t} = \frac{\partial\boldsymbol{b}}{\partial t} + u\frac{\partial\boldsymbol{b}}{\partial x} + v\frac{\partial\boldsymbol{b}}{\partial y} + w\frac{\partial\boldsymbol{b}}{\partial z}\,, \tag{1.9}$$

and for a particular component of $\boldsymbol{b}$, b^x say,

$$\frac{\mathrm{D}b^x}{\mathrm{D}t} = \frac{\partial b^x}{\partial t} + u\frac{\partial b^x}{\partial x} + v\frac{\partial b^x}{\partial y} + w\frac{\partial b^x}{\partial z}\,, \tag{1.10}$$

and similarly for b^y and b^z. In Cartesian tensor notation the expression becomes

$$\frac{\mathrm{D}b_i}{\mathrm{D}t} = \frac{\partial b_i}{\partial t} + v_j\frac{\partial b_i}{\partial x_j} = \frac{\partial b_i}{\partial t} + v_j\partial_j b_i\,, \tag{1.11}$$

where the subscripts denote the Cartesian components, repeated indices are summed, and $\partial_j b_i \equiv \partial b_i/\partial x_j$. In coordinate systems other than Cartesian the advective derivative of a vector is not simply the sum of the advective derivative of its components, because the coordinate vectors themselves change direction with position; this will be important when we deal with spherical coordinates. Finally, we remark that the advective derivative of the position of a fluid element, $\boldsymbol{r}$ say, is its velocity, and this may easily be checked by explicitly evaluating $\mathrm{D}\boldsymbol{r}/\mathrm{D}t$.

1.1.3 Material Derivative of a Volume

The volume that a given, unchanging, mass of fluid occupies is deformed and advected by the fluid motion, and there is no reason why it should remain constant. Rather, the volume will change as a result of the movement of each element of its bounding material surface, and in particular will change if there is a non-zero normal component of the velocity at the fluid surface. That is, if the volume of some fluid is $\int \mathrm{d}V$, then

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\mathbf{S}, \tag{1.12}$$

where the subscript V indicates that the integral is a definite integral over some finite volume V, although the limits of the integral will be functions of time if the volume is changing. The integral on the right-hand side is over the closed surface, S, bounding the volume. Although intuitively apparent (to some), this expression may be derived more formally using Leibniz's formula for the rate of change of an integral whose limits are changing. Using the divergence theorem on the right-hand side, (1.12) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathrm{d}V = \int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V. \tag{1.13}$$

The rate of change of the volume of an infinitesimal fluid element of volume ΔV is obtained by taking the limit of this expression as the volume tends to zero, giving

$$\lim_{\Delta V\to 0}\frac{1}{\Delta V}\frac{\mathrm{D}\Delta V}{\mathrm{D}t} = \nabla\cdot\boldsymbol{v}. \tag{1.14}$$

We will often write such expressions informally as

$$\frac{\mathrm{D}\Delta V}{\mathrm{D}t} = \Delta V\nabla\cdot\boldsymbol{v}, \tag{1.15}$$

with the limit implied.

Consider now the material derivative of some fluid property, ξ say, multiplied by the volume of a fluid element, ΔV. Such a derivative arises when ξ is the amount per unit volume of ξ-substance — the mass density or the amount of a dye per unit volume, for example. Then we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\xi\Delta V) = \xi\frac{\mathrm{D}\Delta V}{\mathrm{D}t} + \Delta V\frac{\mathrm{D}\xi}{\mathrm{D}t}. \tag{1.16}$$

Using (1.15) this becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}(\xi\Delta V) = \Delta V\left(\xi\nabla\cdot\boldsymbol{v} + \frac{\mathrm{D}\xi}{\mathrm{D}t}\right), \tag{1.17}$$

and the analogous result for a finite fluid volume is just

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \xi\,\mathrm{d}V = \int_V\left(\xi\nabla\cdot\boldsymbol{v} + \frac{\mathrm{D}\xi}{\mathrm{D}t}\right)\mathrm{d}V. \tag{1.18}$$

This expression is to be contrasted with the Eulerian derivative for which the volume, and so the limits of integration, are fixed and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \xi\,\mathrm{d}V = \int_V \frac{\partial\xi}{\partial t}\,\mathrm{d}V. \tag{1.19}$$

Now consider the material derivative of a fluid property φ multiplied by the mass of a fluid element, $\rho\Delta V$, where ρ is the fluid density. Such a derivative arises when φ is the amount of φ-substance per unit mass (note, for example, that the momentum of a fluid element is $\rho\boldsymbol{v}\Delta V$). The material derivative of $\varphi\rho\Delta V$ is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}(\varphi\rho\Delta V) = \rho\Delta V\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\frac{\mathrm{D}}{\mathrm{D}t}(\rho\Delta V). \tag{1.20}$$

But $\rho\Delta V$ is just the mass of the fluid element, and that is constant — that is how a fluid element is defined. Thus the second term on the right-hand side vanishes and

$$\frac{\mathrm{D}}{\mathrm{D}t}(\varphi\rho\Delta V) = \rho\Delta V\frac{\mathrm{D}\varphi}{\mathrm{D}t} \qquad \text{and} \qquad \frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi\rho\,\mathrm{d}V = \int_V \rho\frac{\mathrm{D}\varphi}{\mathrm{D}t}\,\mathrm{d}V, \tag{1.21a,b}$$

Material and Eulerian Derivatives

The material derivatives of a scalar (φ) and a vector ($\boldsymbol{b}$) field are given by:

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\nabla\varphi, \qquad \frac{\mathrm{D}\boldsymbol{b}}{\mathrm{D}t} = \frac{\partial\boldsymbol{b}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}. \tag{D.1}$$

Various material derivatives of integrals are:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi\,\mathrm{d}V = \int_V \left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\nabla\cdot\boldsymbol{v}\right)\mathrm{d}V = \int_V \left(\frac{\partial\varphi}{\partial t} + \nabla\cdot(\varphi\boldsymbol{v})\right)\mathrm{d}V, \tag{D.2}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathrm{d}V = \int_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V, \tag{D.3}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\varphi\,\mathrm{d}V = \int_V \rho\frac{\mathrm{D}\varphi}{\mathrm{D}t}\,\mathrm{d}V. \tag{D.4}$$

These formulae also hold if φ is a vector. The Eulerian derivative of an integral is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\,\mathrm{d}V = \int_V \frac{\partial\varphi}{\partial t}\,\mathrm{d}V, \tag{D.5}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\varphi\,\mathrm{d}V = \int_V \frac{\partial\rho\varphi}{\partial t}\,\mathrm{d}V. \tag{D.6}$$

where (1.21b) applies to a finite volume. That expression may also be derived more formally using Leibniz's formula for the material derivative of an integral, and the result also holds when φ is a vector. The result is quite different from the corresponding Eulerian derivative, in which the volume is kept fixed; in that case we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\rho\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}(\varphi\rho)\,\mathrm{d}V. \tag{1.22}$$

Various material and Eulerian derivatives are summarized in the shaded box above.

1.2 THE MASS CONTINUITY EQUATION

In classical mechanics mass is absolutely conserved and in solid-body mechanics we normally do not need an explicit equation of mass conservation. However, in fluid mechanics fluid flows into and away from regions, and fluid density may change, and an equation that explicitly accounts for the flow of mass is one of the equations of motion of the fluid.

1.2.1 An Eulerian Derivation

We will first derive the mass conservation equation from an Eulerian point of view; that is to say, our reference frame is fixed in space and the fluid flows through it.

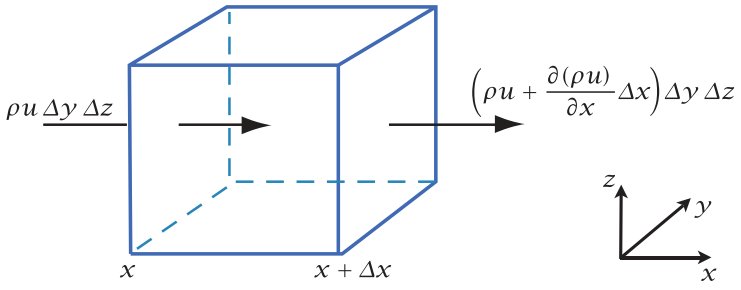

Fig. 1.1 Mass conservation in an Eulerian cuboid control volume. The mass convergence, $-\partial(\rho u)/\partial x$ (plus contributions from the y and z directions), must be balanced by a density increase, $\partial\rho/\partial t$.

Cartesian derivation

Consider an infinitesimal, rectangular cuboid, control volume, $\Delta V = \Delta x\Delta y\Delta z$ that is fixed in space, as in Fig. 1.1. Fluid moves into or out of the volume through its surface, including through its faces in the y–z plane of area $\Delta A = \Delta y\Delta z$ at coordinates x and $x+\Delta x$. The accumulation of fluid within the control volume due to motion in the x-direction is evidently

$$\Delta y\Delta z[(\rho u)(x,y,z) - (\rho u)(x+\Delta x,y,z)] = -\frac{\partial(\rho u)}{\partial x}\bigg|_{x,y,z}\Delta x\,\Delta y\,\Delta z. \tag{1.23}$$

To this must be added the effects of motion in the y- and z-directions, namely

$$-\left[\frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\Delta x\,\Delta y\,\Delta z. \tag{1.24}$$

This net accumulation of fluid must be accompanied by a corresponding increase of fluid mass within the control volume. This is

$$\frac{\partial}{\partial t}\,(\text{density}\times\text{volume}) = \Delta x\,\Delta y\,\Delta z\frac{\partial\rho}{\partial t}, \tag{1.25}$$

because the volume is constant. Thus, because mass is conserved, (1.23), (1.24) and (1.25) give

$$\Delta x\,\Delta y\,\Delta z\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0. \tag{1.26}$$

The quantity in square brackets must be zero and we therefore have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{1.27}$$

This is called the *mass continuity equation* for it recognizes the continuous nature of the mass field in a fluid. There is no diffusion term in (1.27), no term like $\kappa\nabla^2\rho$. This is because mass is transported by the macroscopic movement of molecules; even if this motion appears diffusion-like any net macroscopic molecular motion constitutes, by definition, a velocity field.

Vector derivation

Consider an arbitrary control volume V bounded by a surface S, fixed in space, with by convention the direction of $\boldsymbol{S}$ being toward the outside of V, as in Fig. 1.2. The rate of fluid loss due to flow through the closed surface S is then given by

$$\text{fluid loss} = \int_S \rho\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} = \int_V \nabla\cdot(\rho\boldsymbol{v})\,\mathrm{d}V, \tag{1.28}$$

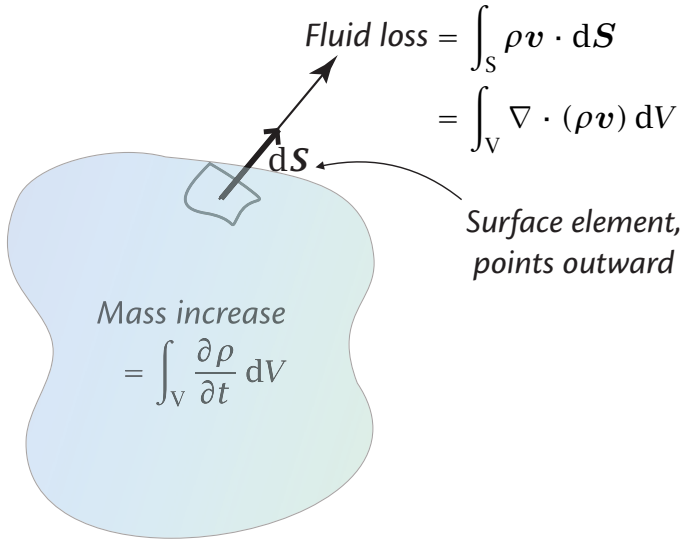

Fig. 1.2 Mass conservation in an arbitrary Eulerian control volume V bounded by a surface S. The mass increase, $\int_V (\partial\rho/\partial t)\, \mathrm{d}V$ is equal to the mass flowing into the volume, $-\int_S (\rho\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = -\int_V \nabla \cdot (\rho\boldsymbol{v})\, \mathrm{d}V$.

using the divergence theorem.

This must be balanced by a change in the mass M of the fluid within the control volume, which, since its volume is fixed, implies a density change. That is

$$\text{fluid loss} = -\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\, \mathrm{d}V = -\int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}V. \tag{1.29}$$

Equating (1.28) and (1.29) yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v})\right] \mathrm{d}V = 0. \tag{1.30}$$

Because the volume is arbitrary, the integrand must vanish and we recover (1.27).

1.2.2 Mass Continuity via the Material Derivative

We now derive the mass continuity equation (1.27) from a material perspective. This is the most fundamental approach of all since the principle of mass conservation states simply that the mass of a given element of fluid is, by definition of the element, constant. Thus, consider a small mass of fluid of density ρ and volume ΔV. Then conservation of mass may be represented by

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho \Delta V) = 0. \tag{1.31}$$

Both the density and the volume of the parcel may change, so

$$\Delta V \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \frac{\mathrm{D}\Delta V}{\mathrm{D}t} = \Delta V \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{v}\right) = 0, \tag{1.32}$$

where the second expression follows using (1.15). Since the volume element is arbitrary, the term in brackets must vanish and

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{v} = 0. \tag{1.33}$$

After expansion of the first term this becomes identical to (1.27). This result may be derived more formally by rewriting (1.31) as the integral expression

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\, \mathrm{d}V = 0. \tag{1.34}$$

Expanding the derivative using (1.18) gives

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\,\mathrm{d}V = \int_V \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v}\right)\mathrm{d}V = 0. \tag{1.35}$$

Because the volume over which the integral is taken is arbitrary the integrand itself must vanish and we recover (1.33). Summarizing, equivalent partial differential equations representing conservation of mass are:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0, \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{1.36a,b}$$

1.2.3 A General Continuity Equation

The derivation of a continuity equation for a general scalar property of a fluid is similar to that for density, except that there may be an external source or sink, and potentially a means of transferring the property from one location to another differently than by fluid motion, for example by diffusion. If ξ is the amount of some property of the fluid per unit volume (the volume concentration, sometimes simply called the concentration), and if the net effect per unit volume of all non-conservative processes is denoted by $Q_{[v,\xi]}$, then the continuity equation for concentration may be written:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\xi\Delta V) = Q_{[v,\xi]}\Delta V. \tag{1.37}$$

Expanding the left-hand side and using (1.15) we obtain

$$\frac{\mathrm{D}\xi}{\mathrm{D}t} + \xi\nabla\cdot\boldsymbol{v} = Q_{[v,\xi]}, \qquad \text{or} \qquad \frac{\partial\xi}{\partial t} + \nabla\cdot(\xi\boldsymbol{v}) = Q_{[v,\xi]}. \tag{1.38}$$

If we are interested in a tracer that is normally measured per unit mass of fluid (which is typical when considering thermodynamic quantities) then the conservation equation would be written

$$\frac{\mathrm{D}}{\mathrm{D}t}(\varphi\rho\Delta V) = Q_{[m,\varphi]}\rho\Delta V, \tag{1.39}$$

where φ is the tracer mixing ratio or mass concentration — that is, the amount of tracer per unit fluid mass — and $Q_{[m,\varphi]}$ represents non-conservative sources of φ per unit mass. Then, since $\rho\Delta V$ is constant we obtain

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = Q_{[m,\varphi]} \qquad \text{or} \qquad \frac{\partial(\rho\varphi)}{\partial t} + \nabla\cdot(\rho\varphi\boldsymbol{v}) = \rho Q_{[m,\varphi]}, \tag{1.40}$$

using the mass continuity equation, (1.36), to obtain the equation on the right. The source term $Q_{[m,\varphi]}$ is evidently equal to the rate of change of φ of a fluid element. When this is so, we often write it simply as $\dot{\varphi}$, so that

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \dot{\varphi}. \tag{1.41}$$

A tracer obeying (1.41) with $\dot{\varphi} = 0$ is said to be *materially conserved*. If a tracer is materially conserved except for the effects of sources or sinks, or diffusion terms, then it is sometimes (if rather loosely) said to be an 'adiabatically conserved' variable, although adiabatic properly means with no heat exchange. If those sources and sinks are in the form of the divergence of a flux with φ satisfying $\rho\mathrm{D}\varphi/\mathrm{D}t = \nabla\cdot\boldsymbol{F}_\varphi$ or equivalently, using the mass continuity equation, $\partial(\rho\varphi)/\partial t + \nabla\cdot(\rho\boldsymbol{v}\varphi) = \nabla\cdot\boldsymbol{F}_\varphi$, then φ is said to be a *conservative* variable because, with no flux boundary conditions, $\int\rho\varphi\,\mathrm{d}V$ = constant. Although momentum as a whole is conserved, momentum is not a materially conserved variable, as we are about to see.