MONOIDAL TOPOLOGY

Monoidal Topology describes an active research area that, after various past proposals on how to axiomatize "spaces" in terms of convergence, began to emerge at the beginning of the millennium. It combines Barr's relational presentation of topological spaces in terms of ultrafilter convergence with Lawvere's interpretation of metric spaces as small categories enriched over the extended real half-line. Hence, equipped with a quantale $\mathcal{V}$ (replacing the reals) and a monad $\mathbb{T}$ (replacing the ultrafilter monad) laxly extended from set maps to $\mathcal{V}$-valued relations, the book develops a categorical theory of $(\mathbb{T}, \mathcal{V})$-algebras that is inspired simultaneously by its metric and topological roots. The book highlights in particular the distinguished role of equationally defined structures within the given lax-algebraic context and presents numerous new results ranging from topology and approach theory to domain theory. All the necessary pre-requisites in order and category theory are presented in the book.

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Monoidal Topology

A Categorical Approach to Order, Metric, and Topology

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To Horst Herrlich

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Preface

Monoidal topology describes an active research area that, after many proposals throughout the past century on how to axiomatize "spaces" in terms of convergence, started to emerge at the beginning of the millennium. It provides a powerful unifying framework and theory for fundamental ordered, metric, and topological structures. Inspired by the topological concept of filter convergence, its methods are lax-algebraic and categorical, with generalized notions of monoid recurring frequently as the fundamental building blocks of its key notions. Since the main components of this new area have to date been available only in a scattered array of research articles, the authors of this book hope that a self-contained and consistent introduction to the theory will serve a broad range of mathematicians, scientists, and their graduate students with an interest in a modern treatment of the mathematical structures in question. With all essential elements from order and category theory provided in the book, it is assumed that the reader will appreciate a framework which highlights the power of equationally defined algebraic structures as particularly important elements of the broader lax-algebraic context which, roughly speaking, replaces equalities by inequalities.

There are two principal roots to the theory presented in this book: Barr's 1970 relational presentation of topological spaces which naturally extends Manes' 1969 equational presentation of compact Hausdorff spaces as the Eilenberg–Moore algebras of the ultrafilter monad, and Lawvere's 1973 description of metric spaces as (small individual) categories enriched over the extended non-negative real half-line. In hindsight, it seems surprising that it took some thirty years until the two general parameters at play here were combined in a compatible fashion, given by a monad $\mathbb{T}$ replacing the ultrafilter monad and a quantale (or, more generally, a monoidal closed category) $\mathcal{V}$ replacing the half-line. Of course, when considered separately, these two pivotal papers triggered numerous important developments. Lawvere's surprising discovery quickly became a cornerstone of enriched category theory, with his characterization of Cauchy completeness in purely enriched-categorical terms enjoying most of the attention, and Barr's paper was followed by at least two major but quite distinct attempts to develop a general topologically inspired theory using a lax-algebraic monad approach, by Manes [1974] and Burroni [1971]. However, the uptake of these articles in terms of follow-up work remained sporadic, perhaps because not many strikingly new applications beyond Barr's work came to the fore, with one prominent exception: the inclusion of Lambek's 1969 multicategories in addition to Barr's topological spaces provides a powerful motivation for Burroni's elegant setting.

In 2000, Bill Lawvere was the first to suggest (in a private communication to Walter Tholen) that, in the same way as topological spaces generalize ordered sets, Lowen's 1989 approach spaces should be describable as generalized metric spaces "using $\mathcal{V}$-multicategories in a good way" instead of just $\mathcal{V}$-categories, thus implicitly envisioning a merger of the parameters $\mathbb{T}$ and $\mathcal{V}$. At about the same time, following a suggestion by George Janelidze, Clementino and Hofmann [2003] gave a lax-algebraic description of approach spaces using a "numerical extension" of the ultrafilter monad. Both suggestions set the stage for Clementino and Tholen [2003] to develop a setting that combines the two parameters efficiently, especially when the monoidal-closed category $\mathcal{V}$ is just a quantale. As emphasized in [Clementino, Hofmann, and Tholen, 2004b], this setting suffices to capture ordered, metric, and topological structures. In a slightly relaxed form, as presented in [Seal, 2005], it also permits to replace ultrafilter convergence by filter convergence (and its "approach generalization") for its key applications, and it is this setting that has been adopted in this book.

When, following a meeting in Barisciano (Italy) in 2006, the authors of this book began to embark decisively on a project to give a self-contained presentation of the emerging theory, the heterogeneous make-up of the group itself made it necessary to document clearly all needed ingredients in a coherent fashion. Hence, this book contains:

- a "crash course" on order and category theory that highlights many aspects not readily available in existing texts and of interest beyond its use for order, metric, and topology;
- an in-depth presentation of the syntactical framework involving the monad $\mathbb{T}$ and the quantale $\mathcal{V}$ needed for a unified treatment of the principal target categories;
- some novel applications leading to new insights, even in the context of ordinary topological spaces, with ample directions to additional or subsequent work that could not be included in this book.

In acknowledging the valuable advice and contributions received from many colleagues, we should highlight first some theses written on subjects pertaining to this book and to various degrees influencing its development, including the Ph.D. theses of Van Olmen [2005], Schubert [2006], Cruttwell [2008], and

Reis [2013], and the Master's theses of Akhvlediani [2008] and Lucyshyn-Wright [2009]. We are grateful especially to Christoph Schubert and Andrei Akhvlediani, who respectively helped to transform Walter Tholen's lecture notes for courses given at the University of Bremen (Germany) in 2003 and at a workshop organized by Francis Borceux at Haute Bodeux (Belgium) in 2007 into something legible and digestible. Christoph was also an active contributor to the various meetings that the group of authors held at the University of Antwerp until 2009, generously organized by Eva Colebunders and Robert Lowen.

The long but surely incomplete list of names of colleagues who offered helpful comments at various stages includes those of Bernhard Banaschewski, Francis Borceux, Franck van Breugel, Marcel Erné, Cosimo Guido, Eraldo Giuli, Horst Herrlich, Kathryn Hess, George Janelidze, Bill Lawvere, Frédéric Mynard, Robert Paré, Hans Porst, Sergejs Solovjovs, Isar Stubbe, Paweł Waszkiewicz, and Richard Wood; we thank them all. We also appreciate the help in proofreading provided by Luca Hunkeler, Valentin Mercier, and Eiichi Piguet.

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We dedicate this book to Horst Herrlich, whose work and dedication to mathematics have had formative influence on all authors of this book.