I

Introduction

Robert Lowen and Walter Tholen

In this introductory chapter we explain, in largely non-technical terms, not only how monoids and their actions occur everywhere in algebra, but also how they provide a common framework for the ordered, metric, topological, or similar structures targeted in this book. This framework is categorical, both at a micro level, since individual spaces may be viewed as generalized small categories, and at a macro level, as we are providing a common setting and theory for the categories of all ordered sets, all metric spaces, and all topological spaces – and many other categories.

Whilst this Introduction uses some basic categorical terms, we actually provide all required categorical language and theory in Chapter II, along with the basic terms about order, metric, and topology, before we embark on presenting the common setting for our target categories. *Many readers may therefore want to jump directly to Chapter III, using the Introduction just for motivation and Chapter II as a reference for terminology and notation.*

I.1 The ubiquity of monoids and their actions

Nothing seems to be more benign in algebra than the notion of *monoid*, i.e. of a set M that comes with an associative binary operation $m : M \times M \longrightarrow M$ and a neutral element, written as a nullary operation $e : 1 \longrightarrow M$. If mentioned at all, normally the notion finds its way into an algebra course only as a brief precursor to the segment on group theory. However, with the advent of monoidal categories, as first studied by Bénabou [1963], Eilenberg and Kelly [1966], Mac Lane [1963], and others, came the realization that monoids and their actions occur everywhere

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in algebra, as the fundamental building blocks of more sophisticated structures. This book is about the extension of this realization from algebra to topology.

I.1.1 Monoids and their actions in algebra

Every algebraist of the past hundred years would subscribe to the claim that free algebras amongst all algebras of a prescribed type contain all the information needed to study these algebras in general. However, what "contain" means was made precise only during the second half of this period. First, there was the observation of the late 1950s [Godement, 1958; Huber, 1961] that the endofunctor $T = GF$ induced by a pair $F \dashv G : \mathsf{A} \to \mathsf{X}$ of adjoint functors comes equipped with natural transformations

$$m : TT \to T \qquad \text{and} \qquad e : 1_{\mathsf{X}} \to T\ ,$$

which, when we trade the Cartesian product of sets and the singleton set 1 for functor composition and the identity functor on X, respectively, are associative and neutral in an easily described diagrammatic sense. Hence, they make T a monoid in the monoidal category of all endofunctors on X, i.e. a *monad* on X [Mac Lane, 1971]. If G is the underlying-set functor of an algebraic category, like the variety of groups, rings, or a particular type of algebras, the free structure TX on X-many generators is just a component of that monad.

On the question of how to recoup the other objects of the algebraic category from the monad they have induced, let us look at the easy example of *actions* of a fixed monoid M in Set. Hence, our algebraic objects are simply sets X equipped with an action $a : M \times X \to X$ making the diagrams

$$\begin{array}{ccc} M\times M\times X & \xrightarrow{1_M\times a} & M\times X \\ {\scriptstyle m\times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ M\times X & \xrightarrow{\ a\ } & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\langle e,1_X\rangle} & M\times X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

commutative. Realizing that $TX = M \times X$ is in fact the carrier of the free structure over X, we may now rewrite these diagrams as

$$\begin{array}{ccc} TTX & \xrightarrow{Ta} & TX \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle a} \\ TX & \xrightarrow{\ a\ } & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle a} \\ & & X\ . \end{array} \tag{I.1.1.i}$$

Using a similar presentation of the relevant morphisms, i.e. of the action-preserving or equivariant maps, Eilenberg and Moore [1965] realized that with every monad $\mathbb{T} = (T, m, e)$ on a category X (in lieu of Set) one may associate the category $\mathsf{X}^{\mathbb{T}}$ whose objects are X-objects X equipped with a morphism $a : TX \to X$ making the two diagrams (I.1.1.i) commutative. Furthermore, there is an adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}} : \mathsf{X}^{\mathbb{T}} \longrightarrow \mathsf{X}$ inducing $\mathbb{T}$, such that, when $\mathbb{T}$ is induced by any adjunction $F \dashv G : \mathsf{A} \longrightarrow \mathsf{X}$, there is a "comparison functor" $K : \mathsf{A} \longrightarrow \mathsf{X}^{\mathbb{T}}$ which, at least for $\mathsf{X} = \mathsf{Set}$, measures the "degree of algebraicity" of A over X. In fact, for any variety of general algebras (with "arities" of operations allowed to be arbitrarily large, as long as the existence of free algebras is guaranteed), K is an equivalence of categories and therefore faithfully recoups the algebras from their monad. By contrast, an application of this procedure to the underlying-set functors of categories of ordered sets or topological spaces in lieu of general algebras would just render the identity monad on Set whose Eilenberg–Moore category is Set itself, i.e. all structural information would be lost.

Whilst all categories of general algebras allowing for free structures may be seen as categories of generalized monoid actions as just described, this fact by no means describes the full extent of the ubiquity of monoids and their actions in algebra. For example, a unital ring R is nothing but an Abelian group R equipped with homomorphisms

$$m : R \otimes R \longrightarrow R \qquad \text{and} \qquad e : \mathbb{Z} \longrightarrow R \,,$$

which are associative and neutral in a quite obvious diagrammatic sense. Hence, when one trades the Cartesian category $(\mathsf{Set}, \times, 1)$ for the monoidal category $(\mathsf{AbGrp}, \otimes, \mathbb{Z})$, monoids R are simply rings, and their actions are precisely the left R-modules. This example, however, is just the tip of an iceberg which places the systematic use of monoidal structures, monoids, and their actions at the core of post-modern algebra.

I.1.2 Orders and metrics as monoids and lax algebras

Although trying to describe ordered sets via the monad induced by the forgetful functor to Set is hopeless, since it induces just the identity monad on Set, a "monoidal perspective" on structures is nevertheless beneficial. First, departing from the notion of a monad, but trading endofunctors T on a category X for relations a on a set X, one can express transitivity and reflexivity of a by

$$a \cdot a \leq a \qquad \text{and} \qquad 1_X \leq a \,, \tag{I.1.2.i}$$

with $\leq$ to be read as set-theoretical inclusion if a is presented as $a \subseteq X \times X$. Hence, with the morphisms $m : a \cdot a \longrightarrow a$ and $e : 1_X \longrightarrow a$ simply given by $\leq$, what we regard as the two indispensable requirements of an *order* a on X, transitivity and reflexivity, are expressed by a carrying the structure of a monoid in the monoidal category of endorelations of X.[1] (The fact that such a relation actually satisfies the equation $a \cdot a = a$ is of no particular concern at this point.)

[1] In this book, in order to avoid the proliferation of meaningless prefixes, we refer to what is usually called a preorder as an order, considering the much less used antisymmetry axiom as an add-on separation condition whenever needed. In fact, with respect to the induced order topology, antisymmetry amounts to the T0-separation requirement.

But it is also possible to consider an order a on X in its role as a structure on X in the spirit of Section I.1.1 as follows. Replacing Set by the category Rel of sets with relations as morphisms and choosing for $\mathbb{T}$ the identity monad on Rel, we see that the inequalities (I.1.2.i) are instances of lax versions of the Eilenberg–Moore requirements (I.1.1.i). Indeed, when formally replacing strict ("=") by lax ("≤") commutativity in (I.1.1.i), we obtain

$$\begin{array}{ccc} TTX & \xrightarrow{Ta} & TX \\ {\scriptstyle m_X}\downarrow & \geq & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow \;\; \leq & \downarrow{\scriptstyle a} \\ & & X\,. \end{array} \tag{I.1.2.ii}$$

In doing so, we suppose that the ambient category X (which is Rel in the case at hand) is ordered, so that its hom-sets are ordered, compatibly with composition. Briefly: ordered sets are precisely the lax Eilenberg–Moore algebras of the identity monad on the ordered category Rel.

Next, presenting relations a on X as functions $a : X \times X \longrightarrow \mathsf{2} = \{\bot < \top\}$ with at most two truth values, let us rewrite the transitivity and reflexivity requirements as

$$a(x, y) \wedge a(y, z) \leq a(x, z) \qquad \text{and} \qquad \top \leq a(x, x)$$

for all $x, y, z \in X$. In this way, there appears a striking formal similarity with what we regard as the two principal requirements of a *metric* $a : X \times X \longrightarrow [0, \infty]$ on X, the triangle inequality and the 0-distance requirement for a point to itself:[2]

$$a(x, y) + a(y, z) \geq a(x, z) \qquad \text{and} \qquad 0 \geq a(x, x)\ .$$

Hence, the set $\mathsf{2}$ with its natural order $\leq$ and its inherent structure $\wedge$ and $\top$ has been formally replaced by the extended real half-line $[0, \infty]$, ordered by the natural $\geq$ (!), and structured by $+$ and 0. Just as for orders, one can now interpret metrics as both monoids and lax Eilenberg–Moore algebras with respect to the identity monad, after extending the relational composition

$$(b \cdot a)(x, z) = \bigvee_{y \in Y} (a(x, y) \wedge b(y, z))$$

for $a : X \times Y \longrightarrow \mathsf{2}$, $b : Y \times Z \longrightarrow \mathsf{2}$ and all $x \in X$, $y \in Y$, by[3]

$$(b \cdot a)(x, z) = \inf_{y \in Y} (a(x, y) + b(y, z))$$

for $a : X \times Y \longrightarrow [0, \infty]$, $b : Y \times Z \longrightarrow [0, \infty]$ and all $x \in X$, $y \in Y$.

[2] Similarly to the use of the term ordered set, in this book we refer to a distance function a satisfying these two basic axioms as a metric, using additional attributes for the other commonly used requirements when needed, like finiteness, symmetry, and separation.

[3] Although we use $\bigwedge, \bigvee$ to refer to infima and suprema in general, in order to avoid ambiguity arising from the "inversion of order" in $[0, \infty]$, we use sup and inf when denoting suprema and infima with respect to the natural order.

The generalized framework encompassing both structures that we will use in this book is provided by a unital *quantale* $\mathcal{V}$ in lieu of $\mathsf{2}$ or $[0, \infty]$; i.e. of a complete lattice equipped with a binary operation $\otimes$ (in lieu of $\wedge$ or $+$) respecting arbitrary joins in each variable, and a $\otimes$-neutral element k (in lieu of $\top$ or 0). The role of the monad $\mathbb{T}$ that appears to be rather artificial in the presentation of ordered sets and metric spaces will become much more pronounced in the presentation of the structures discussed next.

I.1.3 Topological and approach spaces as monoids and lax algebras

In Section I.1.2 we described ordered sets and metric spaces as lax algebras with respect to the identity monad on the category of relations and "numerical" relations, respectively. Taking a historical perspective, we can now indicate how topological spaces fit into this setting once we allow the identity monad to be traded for an arbitrary "lax monad," and how the less-known approach spaces [Lowen, 1997] emerge as the natural hybrid of metric and topology in this context.

Although the axiomatization of topologies in terms of convergence, via filters or nets, has been pursued early on in the development of these structures since Hausdorff [1914], notably by Fréchet [1921] and others, the geometric intuition provided by the open-set and neighborhood perspective clearly dominates the way in which mathematicians perceive topological spaces. Nevertheless, the proof by Manes [1969] that compact Hausdorff spaces are precisely the Eilenberg–Moore algebras of the ultrafilter monad $\mathbb{β} = (\beta, m, e)$ on Set could not be ignored, as it gives the ultimate explanation for why the category $\mathsf{CompHaus}$ behaves in many ways just like algebraic categories do. (For example, just as in algebra, but unlike in the case of arbitrary topological spaces, the set-theoretic inverse of a bijective morphism in $\mathsf{CompHaus}$ is automatically a morphism again.) In this description, a compact Hausdorff space is a set X equipped with a map $a : \beta X \longrightarrow X$ assigning to every ultrafilter χ on X (what turns out to be) its point of convergence in X, requiring the two basic axioms of an Eilenberg–Moore algebra:

$$a(\beta a(\mathcal{X})) = a(m_X(\mathcal{X})) \qquad \text{and} \qquad a(e_X(x)) = x \tag{I.1.3.i}$$

for all $\mathcal{X} \in \beta\beta X$ and $x \in X$; here the following ultrafilters on X are used:

$$e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\}$$

is the principal filter on x;

$$m_X(\mathcal{X}) = \sum \mathcal{X} = \left\{A \subseteq X \mid \{\chi \in \beta X \mid A \in \chi\} \in \mathcal{X}\right\}$$

is the *Kowalsky sum* of $\mathcal{X}$; and

$$\beta a(\mathcal{X}) = a[\mathcal{X}] = \left\{A \subseteq X \mid \{\chi \in \beta X \mid a(\chi) \in A\} \in \mathcal{X}\right\}$$

is simply the image filter of $\mathfrak{X}$ under the map a. Writing $\mathfrak{x} \longrightarrow y$ instead of $a(\mathfrak{x}) = y$ and $\mathfrak{X} \longrightarrow \mathfrak{y}$ instead of $a[\mathfrak{X}] = \mathfrak{y}$, the conditions (I.1.3.i) take the more intuitive form

$$\exists \mathfrak{y} \in \beta X\ (\mathfrak{X} \longrightarrow \mathfrak{y}\ \&\ \mathfrak{y} \longrightarrow z) \iff \textstyle\sum \mathfrak{X} \longrightarrow z \qquad \text{and} \qquad \dot{x} \longrightarrow x$$

for all $\mathfrak{X} \in \beta\beta X$ and $x \in X$. In fact, in the presence of the implication "$\implies$" in the displayed equivalence, the implication "$\impliedby$" comes for free (as $\mathfrak{y} = a[\mathfrak{X}]$ necessarily satisfies $\mathfrak{y} \longrightarrow z$ when $\sum \mathfrak{X} \longrightarrow z$), and conditions (I.1.3.i) take the form

$$\mathfrak{X} \longrightarrow \mathfrak{y}\ \&\ \mathfrak{y} \longrightarrow z \implies \textstyle\sum \mathfrak{X} \longrightarrow z \qquad \text{and} \qquad \dot{x} \longrightarrow x \tag{I.1.3.ii}$$

for all $\mathfrak{X} \in \beta\beta X$, $\mathfrak{y} \in \beta X$, $x, z \in X$.

As Barr [1970] observed, if one allows a to be an arbitrary relation between ultrafilters on X and points of X, rather than a map, so that we are no longer assured that every ultrafilter has a point of convergence (compactness) and that there is at most one such point (Hausdorffness), then the relations $\longrightarrow$ satisfying (I.1.3.ii) describe arbitrary topologies on X, with continuous maps characterized as convergence-preserving maps. Furthermore, given the striking similarity of (I.1.3.ii) with the transitivity and reflexivity conditions of an ordered set, it is not surprising that (I.1.3.ii) gives rise to the presentation of topological spaces as both monoids and lax algebras of the ultrafilter monad.

In this statement, however, we glossed over an important point: having the Set-functor β, one knows what βa is when a is a map, but not necessarily when a is just a relation. Whilst there is a fairly straightforward answer in the case at hand, in general we are confronted with the problem of having to extend a monad $\mathbb{T} = (T, m, e)$ on Set to Rel or, even more generally, to $\mathcal{V}$-Rel, the category of sets and $\mathcal{V}$-relations $r : X \nrightarrow Y$, given by functions $r : X \times Y \longrightarrow \mathcal{V}$. Although for our purposes it suffices that this extension be lax, i.e. quite far from being a genuine monad on $\mathcal{V}$-Rel, the study of the various needed methods of just laxly extending monads on Set to $\mathcal{V}$-Rel can be cumbersome and takes up significant space in this book.

The general framework that emerges as a common setting is therefore given by a unital (but not necessarily commutative) quantale $(\mathcal{V}, \otimes, k)$ and a monad $\mathbb{T} = (T, m, e)$ on Set laxly extended to $\mathcal{V}$-Rel, with the lax extension usually denoted by $\hat{T} : \mathcal{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}$ (although a given $\mathbb{T}$ may have several lax extensions). The lax algebras considered are sets X equipped with a $\mathcal{V}$-relation $a : TX \nrightarrow X$ satisfying the two basic axioms

$$\hat{T}a(\mathfrak{X}, \mathfrak{y}) \otimes a(\mathfrak{y}, z) \le a(m_X(\mathfrak{X}), z) \qquad \text{and} \qquad k \le a(e_X(x), x) \tag{I.1.3.iii}$$

for all $\mathfrak{X} \in TTX$, $\mathfrak{y} \in TX$, $z \in Z$. The lax algebras are to be considered as generalized categories enriched in $\mathcal{V}$, with the domain $\mathfrak{x}$ of the hom-object $a(\mathfrak{x}, y)$

not lying in X but in TX. Furthermore, relational composition can be generalized to *Kleisli convolution* for $\mathcal{V}$-relations $r : TX \times Y \longrightarrow \mathcal{V}$, $s : TY \times Z \longrightarrow \mathcal{V}$ via

$$(s \circ r)(\mathfrak{X}, z) = \bigvee_{\substack{\mathfrak{X} \in TTX \\ m_X(\mathfrak{X}) = \mathfrak{x}}} \bigvee_{\mathfrak{y} \in TY} \hat{T}r(\mathfrak{X}, \mathfrak{y}) \otimes s(\mathfrak{y}, z)$$

for all $\mathfrak{x} \in TX$, $z \in Z$. The lax algebra axioms for (X, a) are then represented via the monoidal structures

$$a \circ a \leq a \qquad \text{and} \qquad 1_X^\sharp \leq a \,,$$

where $1_X^\sharp$ is neutral with respect to the Kleisli convolution.

In this general framework we have so far encountered the objects in the following table, displayed with the corresponding monad $\mathbb{T}$ and quantale $\mathcal{V}$ (here, $\mathsf{P}_+ = (([0, \infty], \geq), +, 0)$ is the extended non-negative real half-line):

$\mathbb{T}$ \ $\mathcal{V}$	$\mathsf{2}$	P_+
Identity monad	ordered sets	metric spaces
Ultrafilter monad	topological spaces	?

Fortunately, the field left blank is filled with a well-studied, but much less familiar, structure, called *approach space*. It is perhaps easiest described in metric terms: an approach structure on a set X can be given by a point-set distance function $\delta : X \times PX \longrightarrow [0, \infty]$ satisfying suitable conditions. A metric space (X, d) becomes an approach space via

$$\delta(x, B) = \inf_{y \in B} d(y, x)$$

for all $x \in X$, $B \subseteq X$. When an approach space is presented as a lax algebra (X, a) with $a : \beta X \times X \longrightarrow [0, \infty]$, one can think of the value $a(\mathfrak{x}, y)$ as the distance that the point y is away from being a limit point of $\mathfrak{x}$. Indeed, a topological space X has its approach structure given by

$$a(\mathfrak{x}, y) = \begin{cases} 0 & \text{if } \mathfrak{x} \longrightarrow y, \\ \infty & \text{otherwise.} \end{cases}$$

As for topological spaces, the more categorical view of approach spaces in terms of convergence proves useful.

I.1.4 The case for convergence

A topology (of open sets) on a set X is most elegantly introduced as a subframe of the powerset X, i.e. a collection of subsets of X closed under finite intersection and arbitrary union. Via complementation, a topology (of closed sets) is equivalently described as a collection closed under finite union and arbitrary intersection, and this simple tool of Boolean duality (switching between open and closed sets) proves to be very useful. There is, however, an unfortunate breakdown of this duality when it comes to morphisms. Although continuous maps are equivalently described by their inverse-image function preserving openness or closedness of subsets, the seemingly most important and natural subclasses of morphisms, namely those continuous maps whose image functions preserve openness or closedness (open or closed continuous maps) behave very differently: whilst open maps are stable under pullback, closed maps are not; not even the subspace restriction $f^{-1}B \to B$ of a closed map $f : X \to Y$ with $B \subseteq Y$ will generally remain closed. Hence, as recognized by Bourbaki [1989], more important than the closed maps are the proper maps, i.e. the morphisms f that are stably closed, so that every pullback of the map f is closed again, also characterized as the closed maps f with compact fibers.

Although under the open- or closed-set perspective no immediate "symmetry" between open and proper maps becomes visible, their characterization in terms of ultrafilter convergence reveals a remarkable duality: a continuous map f : $X \to Y$ is

- *open* if $\mathfrak{y} \longrightarrow f(x)$ (with $x \in X$ and $\mathfrak{y} \in \beta Y$) implies $\mathfrak{y} = f[\mathfrak{x}]$ with $\mathfrak{x} \longrightarrow x$ for some $\mathfrak{x} \in \beta X$,
$$\begin{array}{ccc} \mathfrak{x} & \cdots\!\!\!\rightarrow & x \\ \vdots & & | \\ \mathfrak{y} & \longrightarrow & f(x) \end{array}$$
- *proper* if $f[\mathfrak{x}] \longrightarrow y$ (with $\mathfrak{x} \in \beta X$ and $y \in Y$) implies $y = f(x)$ with $\mathfrak{x} \longrightarrow x$ for some $x \in X$.
$$\begin{array}{ccc} \mathfrak{x} & \cdots\!\!\!\rightarrow & x \\ | & & \vdots \\ f[\mathfrak{x}] & \longrightarrow & y \end{array}$$

In fact, once presented as lax homomorphisms between lax Eilenberg–Moore algebras with respect to the ultrafilter monad (laxly extended from Set to Rel), these two types of special morphisms occur most naturally as the ones for which an inequality characterizing their continuity may be replaced by equality, i.e. by a strict homomorphism condition.

Another indicator why convergence provides a most useful complementary view of topological spaces is the following. For a set X and maps $f_i : X \longrightarrow Y_i$ into topological spaces Y_i, $i \in I$, there is a "best" topology on X making all f_i continuous, often called "weak," but "initial" in this book. Its description in terms of open sets is a bit cumbersome, as it is *generated* by the sets $f^{-1}(B)$, $B \subseteq Y_i$ open, $i \in I$, whereas the characterization in terms of ultrafilter convergence is *immediate*: $\mathfrak{x} \longrightarrow x$ in X precisely when $f_i[\mathfrak{x}] \longrightarrow f(x)$ for all $i \in I$. For example, when $X = \prod_{i \in I} Y_i$ with projections f_i, so that the topology on X just described is the product topology, a proof of the Tychonoff Theorem (on the stability of compactness under products) becomes almost by necessity cumbersome when performed in the open-set environment, but is in fact a triviality in the convergence setting.

We stress, however, the fact that the roles of open sets versus convergence relations are reversed in the dual situation, when one wants to describe the "best" (or "final") topology on a set Y with respect to given maps $f_i : X_i \longrightarrow Y$ originating from topological spaces X_i, $i \in I$. Its description in terms of open sets is immediate, as $B \subseteq Y$ is declared open whenever all $f_i^{-1}(B)$ are open, whereas a characterization in terms of convergence involves a cumbersome generation process.

In conclusion, we regard the two perspectives not at all as mutually exclusive but rather as complementary to each other. Consequently, this book provides a number of results on topological and approach spaces which arise naturally from the general convergence perspective, but which are far from being obvious when expressed in the more classical open-set or point-set distance language.

I.1.5 Filter convergence and Kleisli monoids

To what extent is it possible to trade ultrafilter convergence for filter convergence when presenting topological spaces as in Section I.1.3 or characterizing open and proper maps as in Section I.1.4? In order to answer this question, it is useful to axiomatize topologies on a set X in terms of maps $\nu : X \longrightarrow FX$ into the set FX of filters on X, to be thought of as assigning to each point its neighborhood filter. Ordering such maps pointwise by reverse inclusion and using the same notation as in Section I.1.3, except that now $\circ$ denotes the *Kleisli composition* rather than the Kleisli convolution, one obtains another (and, in fact, more elementary) monoidal characterization of topologies on a set X:

$$\nu \circ \nu \leq \nu \qquad \text{and} \qquad e_X \leq \nu\,;$$

in pointwise terms, this reads as

$$\sum \nu[\nu(x)] \supseteq \nu(x) \qquad \text{and} \qquad \dot{x} \supseteq \nu(x)$$

for all $x \in X$. We say that topological spaces are represented as *Kleisli monoids* (X, ν), or simply as $\mathbb{F}$*-monoids*, since the filter monad $\mathbb{F} = (F, m, e)$ may be traded for any monad $\mathbb{T}$ on Set such that the sets TX carry a complete-lattice order, suitably compatible with the monad operations. As such a monad $\mathbb{T}$ may be characterized via a monad morphism $\tau : \mathbb{P} \longrightarrow \mathbb{F}$, with $\mathbb{P}$ the powerset monad, we call $\mathbb{T}$ *power-enriched*. The basic correspondence between filter convergence and neighborhood systems, given by

$$f \longrightarrow x \iff f \supseteq \nu(x)\,,$$

may now be established at the level of a power-enriched monad $\mathbb{T}$. With a suitable lax extension of $\mathbb{T}$ to Rel, it yields a presentation of $\mathbb{T}$-monoids as lax algebras. For $\mathbb{T} = \mathbb{F}$ it tells us that, remarkably, the characterization (I.1.3.ii) of topological spaces remains valid if we trade ultrafilters for filters. This fact, although established by Pisani [1999] in slightly weaker form, remained unobserved until proved by Seal [2005]. All previous axiomatizations of the notions of topology in terms of filter convergence entailed redundancies.

The answer to our initial question is therefore affirmative with respect to the convergence presentation of topological space. Also, the characterization of open maps given in Section I.1.4 survives the filters-for-ultrafilters exchange, but that of proper maps does not. Hence, we must be cognizant of the fact that the notions introduced for lax algebras will in general depend on the parameters $\mathbb{T}$ and $\mathcal{V}$, not just on the category of lax algebras described by them, such as the category of topological spaces considered here.

I.2 Spaces as categories, and categories of spaces

It has been commonplace since the very beginning of category theory to regard individual ordered sets as categories: they are precisely the categories whose hom-sets have at most one element. By contrast, it was a very bold step for Lawvere [1973] to interpret the distance $a(x, y)$ in a metric space as $\hom(x, y)$. To understand this interpretation, we first recall how ordinary categories fare in the context of orders and metrics as described in Section I.1.2. We then indicate how the consideration of individual ordered sets, metric spaces, topological spaces, and similar objects as small generalized categories leads to new insights and cross fertilization between different areas, as does the investigation of the properties of the category of all such small categories of a particular type.

I.2.1 Ordinary small categories

Replacing "truth values" ($\mathsf{2}$-valued or $[0, \infty]$-valued) by arbitrary sets, for a given set X of "objects" let us consider functions

$$a : X \times X \longrightarrow \mathsf{Set}\,.$$

X is then the set of objects of a category with hom-sets $a(x, y)$ if there are families of maps

$$m_{X,Y,Z} : a(x, y) \times a(y, z) \longrightarrow a(x, z) \qquad \text{and} \qquad e_X : 1 \longrightarrow a(x, x)$$

satisfying the obvious associativity and neutrality conditions, expressible in terms of commutative diagrams. Hence, the notion of small category fits into the same structural pattern already observed for orders and metrics, where now the composition of functions $a : X \times Y \longrightarrow \mathsf{Set}$, $b : Y \times Z \longrightarrow \mathsf{Set}$ is given by

$$(b \cdot a)(x, z) = \textstyle\coprod_{y \in Y} (a(x, y) \times b(y, z))$$

for all $x \in X$, $z \in Z$.

Briefly, if one allows the above-mentioned setting of a unital quantale $(\mathcal{V}, \otimes, k)$ to be extended to that of a monoidal closed category, ordinary small categories