

## Complexity Dichotomies for Counting Problems

### Volume 1: Boolean Domain

Complexity theory aims to understand and classify computational problems, especially decision problems, according to their inherent complexity. This book uses new techniques to expand the theory for use with counting problems. The authors present dichotomy classifications for broad classes of counting problems in the realm of P and NP. Classifications are proved for partition functions of spin systems, graph homomorphisms, constraint satisfaction problems, and Holant problems. The book assumes minimal prior knowledge of computational complexity theory, developing proof techniques as needed and gradually increasing the generality and abstraction of the theory.

This volume presents the theory on the Boolean domain, and includes a thorough presentation of holographic algorithms, culminating in classifications of computational problems studied in exactly solvable models from statistical mechanics.

JIN-YI CAI is Professor of Computer Science and the Steenbock Professor of Mathematical Sciences at the University of Wisconsin–Madison. He was born in Shanghai, China. He studied at Fudan University (class of 77) and at Cornell University, receiving his Ph.D. in 1986. He held faculty positions at Yale University (1986–1989), Princeton University (1989–1993), and SUNY Buffalo (1993–2000), where he rose from Assistant Professor to Full Professor in 1996. He received a Presidential Young Investigator Award (1990), an Alfred P. Sloan Fellowship (1994), and a John Simon Guggenheim Fellowship (1998). He is a Fellow of Association for Computing Machinery (ACM) and American Association for the Advancement of Science (AAAS).

XI CHEN is Associate Professor of Computer Science at Columbia University. He was born in Suzhou, China. He studied at Tsinghua University and received his Ph.D. in 2007. His research focuses on complexity theory and algorithmic game theory. He is the recipient of a National Science Foundation (NSF) CAREER Award, an Alfred P. Sloan Fellowship (2012), and a European Association of Theoretical Computer Science (EATCS) Presburger Award (2015).

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Jin-Yi Cai , Xi Chen

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Volume 1: Boolean Domain

JIN-YI CAI

*University of Wisconsin, Madison*

XI CHEN

*Columbia University, New York, New York*



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## Preface

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The *raison d'être* of computational complexity theory is to understand and classify computational problems according to their inherent complexity. In the past fifteen years or so, there have been enormous advances in the study of the complexity of counting problems. A substantial theory has been developed that classifies every problem in a class of Sum-of-Product computations as being either polynomial-time computable or  $\#P$ -hard. Such theorems are called complexity dichotomies.

This book presents the dichotomy theory for counting problems, expressible as Sum-of-Product computations. More specifically, it presents dichotomy theorems for partition functions of Spin Systems, Graph Homomorphisms, Constraint Satisfaction Problems, and Holant Problems. The theorems have the consequence that, assuming the complexity theory hypothesis  $P \neq \#P$ , which is implied by  $P \neq NP$ , in broad classes of counting problems, up to polynomial-time equivalence, there are only exactly two distinct levels for exact counting complexity.

Volume I presents the theory on the Boolean domain; Volume II presents the theory on general domains. In Volume I a particular feature is a thorough presentation of holographic algorithms, culminating in classifications of computational problems studied in Exactly Solvable Models from statistical mechanics.

Much of the material in the book is presented in book form for the first time. The book assumes minimal prior knowledge of computational complexity theory. The proof techniques are developed as needed. The dichotomy theorems are carefully organized, so that the theory is presented in gradually increasing generality and abstraction.

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Madison, Wisconsin  
New York, New York

Jin-Yi Cai  
Xi Chen