An Introduction to Polynomial and Semi-Algebraic Optimization

This is the first comprehensive introduction to the powerful moment approach for solving global optimization problems (and some related problems) described by polynomials (and even semi-algebraic functions). In particular, the author explains how to use relatively recent results from real algebraic geometry to provide a systematic numerical scheme for computing the optimal value and global minimizers. Indeed, among other things, powerful positivity certificates from real algebraic geometry allow one to define an appropriate hierarchy of semidefinite (sum of squares) relaxations or linear programming relaxations whose optimal values converge to the global minimum. Several specializations and extensions to related optimization problems are also described.

Graduate students, engineers and researchers entering the field can use this book to understand, experiment and master this new approach through the simple worked examples provided.

JEAN BERNARD LASSERRE is Directeur de Recherche at the LAAS-CNRS laboratory in Toulouse and a member of the Institute of Mathematics of Toulouse (IMT). He is a SIAM Fellow and in 2009 he received the Lagrange Prize, awarded jointly by the Mathematical Optimization Society (MOS) and the Society for Industrial and Applied Mathematics (SIAM).
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JEAN BERNARD LASERRE

LAAS-CNRS and Institut de Mathématiques, Toulouse, France
To my daughter Julia, and to Carole ...
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Preface

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Jean B. Lasserre
Symbols

\(\mathbb{N}\), the set of natural numbers
\(\mathbb{Z}\), the set of integers
\(\mathbb{Q}\), the set of rational numbers
\(\mathbb{R}\), the set of real numbers
\(\mathbb{R}_+\), the set of nonnegative real numbers
\(\mathbb{C}\), the set of complex numbers

\(\leq\), less than or equal to
\(\leq\), inequality “\(\leq\)” or equality “\(=\)”

\(A\), matrix in \(\mathbb{R}^{m \times n}\)
\(A_{j}\), column \(j\) of matrix \(A\)
\(A \succeq 0\) (\(\succ 0\)), \(A\) is positive semidefinite (definite)

\(x\), scalar \(x \in \mathbb{R}\)
\(x\), vector \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\)
\(\alpha\), vector \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\)

\(|\alpha| = \sum_{i=1}^{n} \alpha_i\) for \(\alpha \in \mathbb{N}^n\)
\(\mathbb{N}_d^\alpha \subseteq \mathbb{N}^n\), the set \(\{ \alpha \in \mathbb{N}^n : |\alpha| \leq d \}\)
\(x^\alpha\), monomial \(x^\alpha = (x_1^{\alpha_1} \cdots x_n^{\alpha_n})\) \(x \in \mathbb{C}^n\) or \(x \in \mathbb{R}^n\), \(\alpha \in \mathbb{N}^n\)

\(\mathbb{R}[x]\), ring of real univariate polynomials
\(\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]\), ring of real multivariate polynomials

\((x^\alpha), \alpha \in \mathbb{N}^n\), canonical monomial basis of \(\mathbb{R}[x]\)

\(V_C(I) \subseteq \mathbb{C}^n\), the algebraic variety associated with an ideal \(I \subseteq \mathbb{R}[x]\)
\(\sqrt{I}\), the radical of an ideal \(I \subseteq \mathbb{R}[x]\)
\(\sqrt{I}\), the real radical of an ideal \(I \subseteq \mathbb{R}[x]\)

\(I(V_C(I)) \subseteq \mathbb{C}^n\), the vanishing ideal \(\{ f \in \mathbb{R}[x] : f(z) = 0, \forall z \in V_C(I) \}\)
\(V_\mathbb{R}(I) \subseteq \mathbb{R}^n\) (equal to \(V_C(I) \cap \mathbb{R}^n\)), the real variety associated with an ideal \(I \subseteq \mathbb{R}[x]\)

\(I(V_\mathbb{R}(I)) \subseteq \mathbb{R}[x]\), the real vanishing ideal \(\{ f \in \mathbb{R}[x] : f(x) = 0, \forall x \in V_\mathbb{R}(I) \}\)
Symbols

\[ \mathbb{R}[x], \] vector space of real multivariate polynomials of degree at most \( t \)
\[ \sum [x], \] \( \subset \mathbb{R}[x] \), the convex cone of SOS polynomials of degree at most \( 2t \)
\[ \mathbb{R}[x]^*, \] vector space of linear forms on \( \mathbb{R}[x] \)
\[ \mathbb{R}[x]^+_t, \] vector space of linear forms on \( \mathbb{R}[x] \)
\[ y = (y_\alpha), \alpha \in \mathbb{N}^n, \] real moment sequence indexed in the canonical basis of \( \mathbb{R}[x] \)
\[ M_d(y), \] moment matrix of order \( d \) associated with the sequence \( y \)
\[ M_d(g, y), \] localizing matrix of order \( d \) associated with the sequence \( y \) and \( g \in \mathbb{R}[x] \)
\[ P(g) \subset \mathbb{R}[x], \] preordering generated by the polynomials \( (g_j) \subset \mathbb{R}[x] \)
\[ Q(g) \subset \mathbb{R}[x], \] quadratic module generated by the polynomials \( (g_j) \subset \mathbb{R}[x] \)
\[ \text{co} X, \] convex hull of \( X \subset \mathbb{R}^n \)
\[ B(X), \] space of bounded measurable functions on \( X \)
\[ C(X), \] space of bounded continuous functions on \( X \)
\[ M(X), \] vector space of finite signed Borel measures on \( X \subset \mathbb{R}^n \)
\[ \mathcal{M}(X)_+ \subset M(X), \] space of finite (nonnegative) Borel measures on \( X \subset \mathbb{R}^n \)
\[ P(X) \subset \mathcal{M}(X)_+, \] space of Borel probability measures on \( X \subset \mathbb{R}^n \)
\[ L_1(X, \mu), \text{Banach space of functions on } X \subset \mathbb{R}^n \text{ such that } \int_X |f| d\mu < \infty \]
\[ L_\infty(X, \mu), \text{Banach space of measurable functions on } X \subset \mathbb{R}^n \text{ such that } \|f\|_\infty := \text{ess sup } |f| < \infty \]
\[ \sigma(\mathcal{X}, \mathcal{Y}), \text{weak topology on } \mathcal{X} \text{ for a dual pair } (\mathcal{X}, \mathcal{Y}) \text{ of vector spaces} \]
\[ \mu_n \Rightarrow \mu, \text{weak convergence for a sequence } (\mu_n) \subset \mathcal{M}(X)_+ \]
\[ v \ll \mu, \text{is absolutely continuous with respect to } \mu \text{ (for measures)} \]
\[ \uparrow, \text{monotone convergence for nondecreasing sequences} \]
\[ \downarrow, \text{monotone convergence for nonincreasing sequences} \]

SOS, sum of squares
LP, linear programming (or linear program)
SDP, semidefinite programming (or semidefinite program)
GMP, generalized moment problem (or GPM, generalized problem of moments)
SDr, semidefinite representation (or semidefinite representable)
KKT, Karush–Kuhn–Tucker
CQ, constraint qualification
LMI, linear matrix inequality
b.s.a., basic semi-algebraic
b.s.a.l., basic semi-algebraic lifting
l.s.c., lower semi-continuous
u.s.c., upper semi-continuous