1
Infinite Sequences

1.1 Introduction to Sequences

Sequences are useful in the analysis of structures and patterns that occur in a variety of contexts and across a broad range of disciplines. Sequences occur in mathematics, biology, chemistry, and physics as well as in finance, manufacturing, and computer science. A sequence can be used to represent a mathematical structure, a manufacturing process, or the pattern of nucleotides in a DNA molecule. Sequences can also express rules of thumb or general properties of a system. For example, the f-stops 1, 1.4, 2, 2.8, 4, 5.6, 8, 11, and 16 found on the aperture ring of a camera lens essentially form a geometric sequence. This sequence represents the amount of light reaching the camera's film or sensor per unit area. Another example of a practical sequence is the Mariner's Rule of Twelfths – 1, 2, 3, 3, 2, 1 – which is a rule of thumb for estimating water depth when navigating or anchoring a ship in shallow water. Simply stated, a **sequence** is a list of **elements** or **terms** in a particular **order**. The **elements** of a sequence can be numbers, functions, names, letters, and so forth.

What "a particular order" means is probably best demonstrated by way of a simple example. Consider the sequences (A, M, Y) and (M, A, Y). While the two sequences contain the same elements, they are considered to be different sequences, or not **equal**, because the ordering differs.

1.2 Notation

There are a number of ways to represent a sequence. The notation chosen depends on the form of the sequence and what you know about it. One method is to simply **list** the elements of the sequence:

1, 1.4, 2, 2.8, 4, 5.6, 8, 11, 16	(photographic f-stops)
1, 2, 3, 3, 2, 1	(Rule of Twelfths)
1, −1, 1, −1, . . .	(alternating sequence)
2, 4, 6, 8, 10, . . .	(positive even integers)
3, 5, 7, 11, 13, . . .	(prime numbers)
1, 1, 2, 3, 5, 8, . . .	(Fibonacci sequence)

Since the sequence of photographic f-stops and the Rule of Twelfths are examples of **finite sequences**, in that they only contain a finite number of terms. The remaining sequences are examples of **infinite sequences**, where the three little dots at the end indicate that the sequence continues forever.

A sequence may be represented using **index notation**,

$$a_1,\ a_2,\ a_3, \ldots, a_n, \ldots \tag{1.1}$$

where $\boldsymbol{a_1}$ is referred to as the **first term** of the sequence, $\boldsymbol{a_2}$ the **second term,** $\boldsymbol{a_3}$ the **third term,** and $\boldsymbol{a_n}$ the ***n*th term** or the **general term** of the sequence. The **index *n*** indicates the position of the term in the sequence and is typically taken from the set of natural numbers $\{1, 2, 3, 4, \ldots\}$. Index notation is useful when you recognize the pattern or rule generating the sequence, so that the nth or general term of the sequence can be expressed as a formula or a function. Using index notation, the sequence of photographic f-stops, the alternating sequence, and the sequence of positive even integers can be written as follows:

$1, 1.4, 2, \ldots, (\sqrt{2})^8$	(photographic f-stops)
$1, -1, \ldots, (-1)^{n+1}, \ldots$	(alternating sequence)
$2, 4, 6, 8, \ldots, 2n, \ldots$	(positive even integers).

Sequences may also be represented by the notation $\{a_n\}$, where a_n is the nth or general term and it is understood that the index n runs from 1 to ∞ – for example,

$\{(-1)^{n+1}\}$	(alternating sequence)
$\{2n\}$	(positive even integers),

Since the sequence of photographic f-stops is an example of a finite sequence, it is necessary to denote the end of this sequence. In this case it is customary to represent the sequence as:

$$\left\{(\sqrt{2})^n\right\}_{n=0}^{8} \qquad \text{(photographic f-stops)}$$

In this example, a subscript and superscript are used to define the beginning and end of the finite sequence: the indexing begins with $n = 0$ and ends with $n = 8$.

As this example shows, indexing does not necessarily have to begin with the number 1; it can begin and end with any possible integer. The indexing in this example was chosen in order to simplify the expression for the nth term.

It may not always be simple or even possible to find a formula for the general or nth term of a sequences. In such cases, the sequence cannot be represented using index notation, and the elements of the sequence must instead be listed to indicate the sequence. For example, note that the Rule of Twelfths, and the sequence of prime numbers are expressed as a list of elements:

$$\begin{array}{ll} 1, 2, 3, 3, 2, 1 & \text{(Rule of Twelfths)} \\ 1, 1, 2, 3, 5, 8, \ldots & \text{(Fibonacci sequence)} \\ 3, 5, 7, 11, 13, \ldots & \text{(prime numbers)} \end{array}$$

The Fibonacci sequence is an example of a sequence that can be defined **recursively**. A **recurrence relation** is an expression that relates the nth element of a sequence to a previous element or elements. The **recurrence relation** for the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... is expressed as follows:

$$a_1 = a_2 = 1;$$
$$a_{n+2} = a_n + a_{n+1}.$$

In Chapter 4, we will develop the necessary tools to find a general expression for the terms of the Fibonacci sequence, and we will then specify the sequence using index notation.

The sequence of prime numbers 2, 3, 5, 7, 11, ... is intriguing because there is no known formula capable of generating all the prime numbers. Therefore, we are reduced to presenting the prime numbers as a list. The distribution of prime numbers is currently an open question in mathematics for which there is a related prize, the Clay Mathematics Institute Millennium Prize.

1.3 Example Sequences

Arithmetic, harmonic, and geometric sequences are three types of sequences that are easily defined because there is a constant relation between consecutive terms.

1.3.1 Arithmetic Sequences

An **arithmetic sequence** is a sequence in which consecutive terms differ by a constant amount called the **common difference**. If the first term of the sequence is $\boldsymbol{a}$ and the common difference is $\boldsymbol{d}$**,** then the arithmetic sequence is represented by

$$a,\ a+d,\ a+2d,\ 1+3d, \ldots,\ a+(n-1)d, \ldots \tag{1.2}$$

Each term of the sequence can be obtained by adding the common difference d to the previous term – for example,

$$\begin{array}{l} 1,\ 2,\ 3,\ 4, \ldots \\ 3,\ 7,\ 11,\ 15, \ldots \\ 10,\ 7,\ 4,\ 1, -2, -5, \ldots \end{array}$$

are arithmetic sequences with the common differences 1, 4, and −3, respectively.

1.3.2 Harmonic Sequences

The terms of a **harmonic sequence** are the reciprocals of the terms of an arithmetic sequence, so a harmonic sequence is expressed as

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots, \frac{1}{a+(n-1)d}, \ldots \tag{1.3}$$

Using the arithmetic sequences given above, we can form the corresponding harmonic sequences:

$$\begin{array}{l} 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \\ \frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \ldots \\ \frac{1}{10}, \frac{1}{7}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{15}, \ldots \end{array}$$

A physical example of the reciprocal relationship between arithmetic and harmonic sequences is the reciprocal relationship between wavelength and frequency:

$$\lambda \propto \frac{1}{f}.$$

As an example, consider the strings of a musical instrument. As the strings are fixed at both ends, the longest standing wave, or the fundamental mode supported by such a vibrating string, has a wavelength λ that is twice the length of the string. This fundamental wavelength consists of a round trip along the string, with a half-cycle fitting between the nodes at the ends of the string. The other vibrational modes (or harmonics) supported by the string occur at $\lambda/2$,

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$\lambda/3$, $\lambda/4$, . . . When expressed in terms of the wavelength, the vibrational modes supported by the string form the following sequence:

$$\lambda,\ \frac{\lambda}{2},\ \frac{\lambda}{3},\ \frac{\lambda}{4}, \ldots,\ \frac{\lambda}{n}, \ldots,$$

which by our definition is a harmonic sequence in which the common difference d is 1. We could also choose to characterize the vibrational modes of the string in terms of their frequency rather than their wavelength. If the fundamental mode of the string vibrates with frequency f, then the higher harmonic modes are found at the frequencies $2f$, $3f$, $4f$, . . ., which form the arithmetic sequence

$$f, 2f, 3f, 4f, \ldots, nf, \ldots$$

Note that the reciprocal of each term nf of the arithmetic sequence is $1/nf$, which can be rewritten as λ/n using the inverse relationship ($\lambda \propto 1/f$) between wavelength and frequency, thereby demonstrating that the terms of a harmonic sequence are the reciprocals of the terms of an arithmetic series.

1.3.3 Geometric Sequences

Geometric sequences occur in many different contexts and appear in problems involving growth or decay. In biology, this may be the growth or decay of a population; in physics, it may be the change in the number of particles due to a chain reaction or decay; and in finance, it may be the change in the value of an account due to interest.

A **geometric sequence** is a sequence in which there is a constant ratio between consecutive terms; this ratio is referred to as the **common ratio**. Each term of the sequence can be obtained by multiplying the previous term by the common ratio. If the first term of the geometric sequence is $\boldsymbol{a}$ and the common ratio is $\boldsymbol{r}$, then the geometric sequence is written as

$$a,\ ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots \tag{1.4}$$

– for example,

$$1, 1.4, 2, \ldots, (\sqrt{2})^n, \ldots \qquad \text{(the photographic f-stops)}$$
$$6, -3, \frac{3}{2}, \frac{-3}{4}, \ldots$$
$$2, 6, 18, 54, \ldots$$

are geometric sequences with common ratios of $\sqrt{2}$, $-1/2$, and 3, respectively. The common ratio of a geometric sequence is found by taking the ratio of consecutive terms:

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_{n+1}}{a_n} = \ldots \tag{1.5}$$

A simple physical example of a geometric sequence is the decreasing height of successive bounces of a ball. Consider an experiment in which a ball is dropped onto a hard surface [1]. The ball is initially dropped from the height $H_0 = 40.5$ cm, and the maximum height of each successive bounce is recorded: $H_0 = 40.5$ cm, $H_1 = 37.0$ cm, $H_2 = 34.5$ cm, $H_3 = 32.3$ cm, $H_4 = 30.24$ cm, $H_5 = 28.2$ cm, and $H_6 = 26.4$ cm. The heights of successive bounces thus form the following sequence:

$$40.5,\ 37.0,\ 34.5,\ 32.3,\ 30.2,\ 28.2,\ 26.4.$$

To recognize that this sequence of successive heights is a geometric sequence, you need to recognize that there is a **common ratio** between consecutive terms. Recall that for a geometric sequence, the common ratio between consecutive terms is given by

$$r = \frac{a_{n+1}}{a_n}.$$

Using the ratios of consecutive heights, we would find that

$$\frac{H_1}{H_0} = \frac{H_2}{H_1} = \frac{H_3}{H_2} = \frac{H_4}{H_3} = \frac{H_5}{H_4} = \frac{H_6}{H_5} = \text{constant} = r.$$

A few strokes on a calculator will confirm that a common ratio exists: $r \approxeq 0.9$ (to the first decimal place). Therefore, the nth or general term of the sequence of bounce heights is given by

$$H_n = H_0(0.9)^n.$$

Another example of a geometric sequence is as follows. A chemistry instructor once told me that 99% of an unwanted solution can be rinsed from a container if the container is filled with water and emptied out three times in a row. Is this statement reasonable? If we let a represent the original amount of unwanted solution in the container and let r represent the percentage of fluid that is retained in the container when it is emptied out (i.e., the percentage of fluid that is left clinging to the interior of the container), we find that

$ar^0 = a$ = original amount of unwanted solution,
ar = amount of solution remaining in container after filling with water and emptying,
ar^2 = amount of solution remaining in container after second rinsing, and
ar^3 = amount of solution remaining in container after third rinsing.

Notice that the consecutive terms *a, ar,* ar^2, ar^3 form a geometric sequence. If the container is to be 99% clean, as claimed by the chemistry instructor, this implies that only 1% of the original solution remains in the container after three rinses. Expressed in terms of the geometric progression, the statement implies that

$$\frac{ar^3}{a} = 1\% = 0.01.$$

Solving for r, we find that

$$r^3 = 0.01$$

and thus that

$$r = 0.215 = 21.5\%.$$

So even if the container isn't completely emptied out during each rinse but retains some amount of the unwanted solution (up to 21.5% in this case), the amount of unwanted fluid remaining in the container is given by

$$a_n = a_0(0.215)^n,$$

and the amount of unwanted solution that has been removed will approach 99% after three rinses.

1.4 Limits and Convergence

If the consecutive terms of a sequence approach a constant or limiting value, the sequence is said to **converge** (or be **convergent**). If a sequence does not converge, then it is said to **diverge** (or be **divergent)**. In many applications, it is necessary to determine whether a particular infinite sequence is convergent or divergent. It may also be necessary to determine the limiting value of a convergent sequence. In this section, we will develop the concept of convergence and introduce methods for identifying convergence and for determining the limiting value of a convergent sequence.

Put simply, a convergent sequence is one in which the consecutive elements of the sequence get arbitrarily close to some value. For example, the terms of the sequence

$$0.9, 0.99, 0.999, 0.9999, \ldots$$

can be observed to approach one (the limit of the sequence), so this sequence is convergent. More formally, an infinite sequence has a **limit** if the nth or general term a_n converges to some constant L as n becomes very large:

$$\lim_{n\to\infty} a_n = L.$$

If L is a real number, the sequence is said to **converge to** L.

If the successive terms of a sequence do not approach a limit, the sequence is divergent. A straightforward example of a divergent sequence would be a sequence whose nth term becomes arbitrarily large in magnitude as n approaches infinity:

$$\lim_{n\to\infty} a_n = \pm\infty.$$

For example, the sequence of positive integers

$$1,\ 2,\ 3,\ 4\ldots$$

diverges. A more subtle example of divergence appears in the infinite sequence

$$\{1, -1, 1, -1, 1\ldots\}.$$

The general expression for the sequence is $a_n = (-1)^n$. Taking the limit of the general expression as n gets large, we find that the nth term does not approach a constant value; rather, the sequence oscillates between positive and negative 1. By definition, a convergent sequence has only one limit, but the sequence $\{1, -1, 1, -1, 1 \ldots\}$ does not approach a single limit and is therefore divergent. We will revisit this example momentarily, after we develop a more precise definition of convergence.

The more precise, definition of convergence is as follows: an infinite sequence $\{a_{\text{n}}\}$ converges to a limit L if for every $\varepsilon > 0$, no matter how small, there exists a positive number $N > 0$ such that for all $n > N$, a_n remains arbitrarily close to L (i.e., $|a_n - \text{L}| < \varepsilon$). If the limit L does not exist, then the sequence diverges. Figure 1.1 illustrates this definition of convergence.

For a divergent sequence, the condition $|a_n - \text{L}| < \varepsilon$ cannot be fulfilled, even for very large n. Consider the previous example: $\{1, -1, 1, -1, \ldots\}$. For very large n, the general term a_n is ±1, and the condition $|\pm1 - \text{L}| < \varepsilon$ cannot be fulfilled for an arbitrarily small ε. Because the consecutive terms of the sequence do not approach a single value, a limit does not exist, and the sequence is divergent.

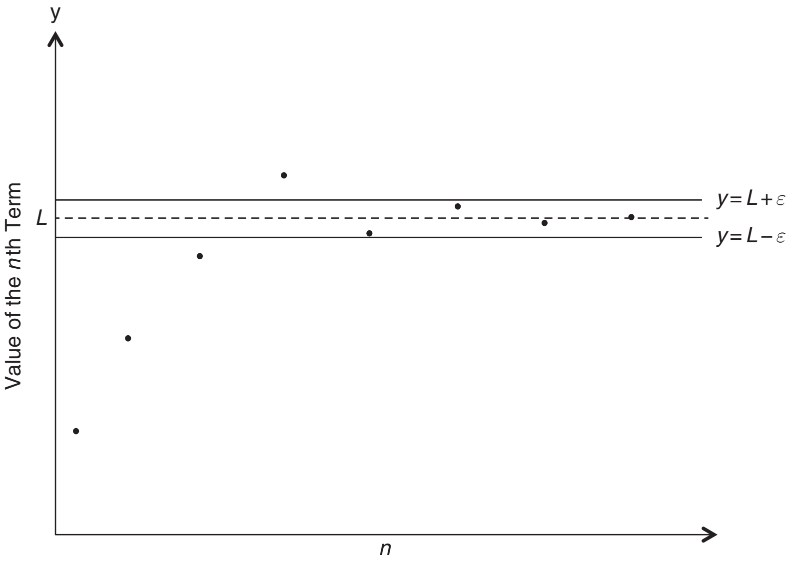

Figure 1.1 Illustration of the convergence of a sequence to the limit L.

To establish the convergence of a sequence, we need to be able to prove convergence by either resorting to the definition of convergence or being able to take the limit of the nth term or general term of the sequence. In many cases, taking the limit of the nth term will be a straightforward process. For example, consider the harmonic and arithmetic sequences representing the vibrational modes of a fixed string. Recall that we developed two sequences in Sections 1.3.1 and 1.3.2:

$$\lambda, \frac{\lambda}{2}, \frac{\lambda}{3}, \frac{\lambda}{4}, \ldots, \frac{\lambda}{n}, \ldots = \left\{\frac{\lambda}{n}\right\} \qquad \text{(sequence of allowed wavelengths)}$$

and

$$f, 2f, 3f, 4f, \ldots nf, \ldots = \{nf\} \qquad \text{(sequence of allowed frequencies).}$$

Taking the limit of the nth term of the harmonic sequence of allowed wavelengths, we find that

$$\lim_{n\to\infty} \frac{\lambda}{n} \to \frac{\lambda}{\infty} \to 0.$$

So the sequence of allowed wavelengths is convergent, and it converges to zero. Taking the limit of the general term of the sequence of allowed frequencies, we find that

$$\lim_{n\to\infty} nf \to \infty f \to \infty,$$

and therefore the sequence diverges. In both of these examples, the limit was very easy to evaluate, as no algebraic manipulation was required. Let us now consider a slightly more complicated example, namely, the infinite sequence

$$\left\{\frac{5n^3}{n^3 + 3\sqrt{4 + n^6}}\right\}.$$

To test this sequence for convergence, we simply need to take the limit of the general term:

$$\lim_{n\to\infty} \frac{5n^3}{n^3 + 3\sqrt{4 + n^6}}.$$

Dividing the numerator and denominator by n^3 and taking the limit, we find that

$$\lim_{n\to\infty} \frac{5}{1 + 3\sqrt{\frac{4}{n^6} + 1}} \to \frac{5}{1+3} = \frac{5}{4},$$

and so the sequence converges to the limit 5/4.

Occasionally, there are situations where it may be necessary to resort to the definition of convergence in order to establish the convergence of a sequence. As an example, we will use the definition of convergence to prove that the geometric sequence $\{r^n\}$ converges to the limit for constant r, where $|r| < 1$ [2]. Using the definition of convergence, we need to show that for every $\varepsilon > 0$, no matter how small, there exists a positive number $N > 0$ such that for all $n > N$, r^n remains arbitrarily close to zero (i.e., $|r^n - 0| < \varepsilon$). For $r = 0$, it is obvious that the sequence $\{r^n\}$ is constant $(0, 0, 0, \ldots)$; the limit is therefore 0, and the sequence converges. For the case where $0 < |r| < 1$, we need to show that there exists a positive number N such that for all $n > N$, $|r^n - 0| < \varepsilon$, no matter how small ε is. We can rewrite the inequality

$$|r^n - 0| < \varepsilon$$

as

$$|r|^n < \varepsilon.$$

Taking the logarithm of the inequality, we find that

$$n \ln|r| < \ln \varepsilon.$$