

Twelve Landmarks of Twentieth-Century Analysis

The striking theorems showcased in this book are among the most profound results of twentieth-century analysis. The authors' original approach combines rigorous mathematical proofs with commentary on the underlying ideas to provide a rich insight into these landmarks in mathematics. Results ranging from the proof of Littlewood's conjecture to the Banach–Tarski paradox have been selected for their mathematical beauty as well as their educative value and historical role. Placing each theorem in historical perspective, the authors paint a coherent picture of modern analysis and its development, whilst maintaining mathematical rigour with the provision of complete proofs, alternative proofs, worked examples, and more than 150 exercises and solution hints.

This edition extends the original French edition of 2009 with a new chapter on partitions, including the Hardy–Ramanujan theorem, and a significant expansion of the existing chapter on the corona problem.

Twelve Landmarks of Twentieth-Century Analysis

D. CHOIMET

Lycée du Parc, Lyon

H. QUEFFÉLEC

Université de Lille

Illustrated by

MICHAËL MONERAU

Translated from the French by

DANIÈLE GIBBONS and GREG GIBBONS



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 D. Choimet, H. Queffélec
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to our students

Contents

<i>Foreword</i>	<i>page xi</i>
<i>Preface</i>	<i>xiii</i>
1 The Littlewood Tauberian theorem	1
1.1 Introduction	1
1.2 State of the art in 1911	7
1.3 Analysis of Littlewood’s 1911 article	10
1.4 Appendix: Power series	27
Exercises	31
2 The Wiener Tauberian theorem	39
2.1 Introduction	39
2.2 A brief overview of Fourier transforms	41
2.3 Wiener’s original proof	44
2.4 Application to Littlewood’s theorem	57
2.5 Newman’s proof of the Wiener lemma	61
2.6 Proof of Wiener’s theorem using Gelfand theory	63
Exercises	66
3 The Newman Tauberian theorem	73
3.1 Introduction	73
3.2 Newman’s lemma	74
3.3 The Newman Tauberian theorem	79
3.4 Applications	83
3.5 The theorems of Ikehara and Delange	93
Exercises	99
4 Generic properties of derivative functions	103
4.1 Measure and category	103
4.2 Functions of Baire class one	105

4.3	The set of points of discontinuity of derivative functions	107
4.4	Differentiable functions that are nowhere monotonic	112
	Exercises	116
5	Probability theory and existence theorems	120
5.1	Introduction	120
5.2	Khintchine's inequalities and applications	121
5.3	Hilbertian subspaces of $L^1([0, 1])$	132
5.4	Concentration of binomial distributions and applications	134
	Exercises	143
6	The Hausdorff–Banach–Tarski paradoxes	148
6.1	Introduction	148
6.2	Means	151
6.3	Paradoxes	162
6.4	Superamenability	173
6.5	Appendix: Topological vector spaces	176
	Exercises	177
7	Riemann's "other" function	182
7.1	Introduction	182
7.2	Non-differentiability of the Riemann function at 0	184
7.3	Itatsu's method	185
7.4	Non-differentiability at the irrational points	191
	Exercises	212
8	<i>Partitio numerorum</i>	219
8.1	Introduction	219
8.2	The generating function	226
8.3	The Dedekind η function	227
8.4	An equivalent of $p(n)$	241
8.5	The circle method	248
8.6	Asymptotic developments and numerical calculations	259
8.7	Appendix: Calculation of an integral	261
	Exercises	263
9	The approximate functional equation of the function θ_0	267
9.1	The approximate functional equation	268
9.2	Other forms of the approximate functional equation and applications	275
	Exercises	286

Contents

ix

10 The Littlewood conjecture	292
10.1 Introduction	292
10.2 Properties of the L^1 -norm and the Littlewood conjecture	298
10.3 Solution of the Littlewood conjecture	303
10.4 Extension to the case of real frequencies	312
Exercises	325
11 Banach algebras	329
11.1 Spectrum of an element in a Banach algebra	330
11.2 Characters of a Banach algebra	333
11.3 Examples	338
11.4 C^* -algebras	342
Exercises	346
12 The Carleson corona theorem	353
12.1 Introduction	353
12.2 Prerequisites	354
12.3 Beurling's theorem	363
12.4 The Lagrange–Carleson problem for an infinite sequence	367
12.5 Applications to functional analysis	382
12.6 Solution of the corona problem	391
12.7 Carleson's initial proof and Carleson measures	412
12.8 Extensions of the corona theorem	417
Exercises	420
13 The problem of complementation in Banach spaces	429
13.1 Introduction	429
13.2 The problem of complementation	431
13.3 Solution of problem (9)	436
13.4 The Kadeč–Snobar theorem	438
13.5 An example “à la Liouville”	443
13.6 An example “à la Hermite”	445
13.7 More recent developments	449
Exercises	452
14 Hints for solutions	460
Exercises for Chapter 1	460
Exercises for Chapter 2	461
Exercises for Chapter 3	464
Exercises for Chapter 4	465
Exercises for Chapter 5	468
Exercises for Chapter 6	469
Exercises for Chapter 7	472

Exercises for Chapter 8	475
Exercises for Chapter 9	475
Exercises for Chapter 10	477
Exercises for Chapter 11	479
Exercises for Chapter 12	481
Exercises for Chapter 13	485
<i>References</i>	489
<i>Notations</i>	497
<i>Index</i>	500

Foreword

Analysis. . . the word is dangerous. Mention it at a dinner party, and depending on your guests, it will bring to mind lab coats and test tubes, or couches and psychoanalysts, or perhaps again those experts that unveil the subtleties of an economical or political crisis. Clarify that you are referring to mathematical analysis and the image will change: former students will then recall memories of derivatives and integrals, and no doubt remind you that it was much easier to calculate the former than the latter. . . But perhaps one might ask you: Mathematical analysis, no doubt it's all very nice, but what's its point? In fact, what are you analysing?

The book of Denis Choimet and Hervé Queffélec provides brilliant and profound answers to these questions in a most agreeable manner. We follow the evolution of analysis throughout the twentieth century, from the founding fathers Hardy and Littlewood, to the creators of spaces Wiener and Banach and up through contemporaries such as Lennart Carleson. The historical perspective helps us understand the motivation behind the problems, and the naturalness of their solutions. Moreover, analysis is shown clearly for what it is: a discipline situated in the heart of mathematics, indissolubly linked to arithmetic and number theory, to combinatorics, to probability theory, to logic, to geometry. . . Its objective is hence to serve mathematics and consequently all of the sciences, and thereby each and every one of us.

I have the pleasure of knowing Denis and Hervé. Hence I can assert that their knowledge of analysis can be qualified as encyclopaedic. However, they were not attempting to write an encyclopaedia, and the roots of their work can be found more in Cambridge than in Paris, Warsaw or Moscow. This wise approach allowed them to explore multiple directions right up to the most recent results, while maintaining the profound unity of a very reasonably formatted book, providing constant encouragement to the reader.

Reaching the end, the reader will lay down the book (close at hand, because there are works that demand to be re-read) with the satisfaction of now having a better understanding of analysis. He will also wish to congratulate Denis Choimet and Hervé Queffélec for their collaboration, which illustrates the connectedness of mathematics and of the community of mathematicians. Whether we study them in “classes préparatoires” or in a university, the mathematics stay equally fascinating. Let us not disfigure them by zebra-stripping boundaries.

But time for a break from lyricism, to make way for mathematics. Happy reading to all! You are in for a real treat.

Gilles Godefroy, September 2014.

Preface

This book has a history: it was born after the encounter of two professors from different generations, on the occasion of a series of mathematics seminars organised by the younger of the two at the Lycée Clemenceau in Nantes, in the early part of the years 2000 onwards. The prime objective of these seminars was to allow the professors of this establishment to keep a certain mathematical awareness that the sustained rhythm of preparing students for competitive entrance exams did not always facilitate. The seminars took place roughly once a month, and lasted an hour and a half. Over the years, the professors were joined by an increasing number of students from their classes; a vocation for mathematics was born for many of these, possibly in part due to this initiative. Both authors gave half a dozen talks at these seminars, on themes of their choosing, with a strong emphasis (but not exclusively) on classical analysis.

After the nomination of one of us to Lyon, we thought it would be interesting to assemble and write up these talks in more detail, and to find a connection between them. It seemed to us that a good starting point would be the 1911 paper of Littlewood (Chapter 1), which is at the same time the founding point of what we today call Tauberian theorems, and the beginning of the famous collaboration between Hardy and Littlewood that spanned 35 years, until Hardy's death in 1947. This collaboration produced a large number of remarkable discoveries, not the least of which was that of Ramanujan. The magnificent work of Hardy and Ramanujan on the asymptotic behaviour of the partition function is in fact the subject of an entire chapter (Chapter 8).

Some of these discoveries are explained in detail, in addition to the converse of Abel's radial theorem (Chapter 1) – from the functional equation (approximated or not) of the Jacobi θ_0 function and its applications – via Diofantine approximations and continued fractions, to exponential sums and the close study of the “other” function of Riemann (Chapters 7 and 9), and in

passing the asymptotic behaviour of the partition function (Chapter 8). Important extensions of this work include Wiener's Tauberian theorem (Chapter 2), the Tauberian theorems of Ikehara and Newman (Chapter 3), which are precursors of Banach algebras and Gelfand theory (Chapter 11), the latter giving rise to the corona problem, brilliantly solved by Carleson in 1962, shortly after his characterisation of interpolation sequences (Chapter 12). Beurling, the supervisor of Carleson (among others), provided a description of the invariant subspaces under the shift operator, a jewel of functional analysis of the twentieth century: this completes the long Chapter 12. Another extension of the study of sums of squares in Chapter 9 is the Littlewood conjecture about the L^1 -norm of exponential sums, which was only resolved in 1981 (Chapter 10). A good half of the book thus pays tribute to the English school of analysis and, in passing, to the Swedish and American schools.

A second main theme starts from the work of the Polish school in the 1930s, in particular that of Stefan Banach. The spaces that today are given his name have been the subject of innumerable studies; one of their specific properties, complementation, is described in Chapter 13. This school highly prized the works of the French, in particular those of Baire and Lebesgue. These in turn are well represented in this book, through the study of the generic properties of derivative functions (Chapter 4), generic properties in the domain of probability theory (Chapter 5, which acknowledges the contributions of the Russian school, with Kolmogorov and Khintchine), and finally the paradoxical properties in measure theory (Chapter 6 on the paradox of Banach–Tarski).

All of these works are profound and difficult, but they deserve to be better known and popularised throughout the mathematical community, both from an historical and a scientific point of view. This has been our ambition.

A few words on the style of this book: we did not seek to write a text for highly skilled specialists, thus we were not ashamed to provide many reminders and lots of details and heuristic explanations, and to provide an historical perspective. Nor did we try to write a book for skilled generalists, thus we were not ashamed to provide complete and rigorous proofs, even if very difficult. Therefore, depending on the themes we study, our book spans multiple levels: some portions are at a graduate level, others are at an advanced undergraduate level, the average being somewhere between the two. We thought it useful to extend each chapter with a dozen or so exercises, as a complement to the main text or as an incentive for the reader to continue his reflections. These exercises do not have detailed solutions, but we hope to have provided sufficient references and hints for a reasonably courageous and interested reader to tackle them.

Preface

xv

We hope that this book will serve a large audience, even if only now and then: we are thinking of our colleagues, as well as graduate students or *amateurs* of mathematics and beauty (*amateurs* is to be understood as Jean-Pierre Kahane would say).

Our thanks go to our friend Rached Mneimné, whose enthusiasm, openness and efficiency allowed this atypical book to be published, and to Gilles Godefroy, who was kind enough to write a friendly foreword. We sincerely thank our colleagues and former students who accepted reading in depth certain chapters and providing us with precise and constructive feedback: Walter Appel, Frédéric Bayart, Nicolas Bonnotte, Rémi Catellier, Vincent Clapiès, Jean-François Deldon, Quentin Dufour, Jordane Granier, Jérémy Guéré, Denis Jourdan, Xavier Lamy, Stéphane Malek, Thomas Ortiz, Marc Pauly, Michel Staïner, Carl Tipler. We also address warm thanks to the staff at CUP who trusted us and helped us with great kindness and professionalism during the final steps of this translation: Emma Collison, Clare Dennison, Katherine Law, Roger Astley.

Special thanks must be addressed to Bruno Calado and Michaël Monerau. Bruno proofread (in record time) the whole of the manuscript, flushed out an incalculable number of misprints, and proposed a multitude of interesting improvements: without him, the book would not be as polished as we wanted. Michaël not only read in detail several chapters, but also provided 20 or so magnificent diagrams which greatly help in reading and understanding the underlying text, at times extremely technical. It was truly to our benefit that they put their competence at our service.

Last but not least, we could have contemplated translating this new edition by ourselves, producing no doubt masses of “Frenghish”. In any case, we totally underestimated the effort involved in translating 500 pages of serious mathematics! As luck would have it, we were introduced to Danièle and Greg Gibbons, both fluent in French, English and . . . mathematics, thus able not only to read our text, but to understand what we were talking about. They did an enormous and excellent job of translating; and for the choice of certain technical terms, it was a true collaboration and a pleasure to discuss with them. Our warmest thanks go to them here.

We welcome with great interest your remarks.