THE LOGIC OF INFINITY

Few mathematical results capture the imagination like Georg Cantor's groundbreaking work on infinity in the late nineteenth century. This opened the door to an intricate axiomatic theory of sets, which was born in the decades that followed.

Written for the motivated novice, this book provides an overview of key ideas in set theory, bridging the gap between technical accounts of mathematical foundations and popular accounts of logic. Readers will learn of the formal construction of the classical number systems, from the natural numbers to the real numbers and beyond, and see how set theory has evolved to analyse such deep questions as the status of the Continuum Hypothesis and the Axiom of Choice. Remarks and digressions introduce the reader to some of the philosophical aspects of the subject and to adjacent mathematical topics. The rich, annotated bibliography encourages the dedicated reader to delve into what is now a vast literature.

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Barnaby Sheppard
Frontmatter
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Audience

This book is to be carefully enclosed in a time machine and sent to a much younger version of myself, to whom it is addressed: a hopeless daydreamer already intoxicated with mathematics, but perhaps not yet fully entangled in the wheels of a university education. Having taught myself all the mathematics I could find in a pre-internet era, I was aware that a whole universe of new ideas was out there waiting to be discovered. I would have opened this book, read parts of it in a peculiar order, and eventually completed all of it after filling a few notebooks with mathematical sketches and questions. It would have captured my imagination and made me want to learn more, so I can only hope that there are others who might also find something of interest herein.

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Preface

> *No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.*
>
> – David Hilbert[1]

In the later years of the nineteenth century Georg Cantor discovered that there are different sizes of infinity. What had begun as the study of a concrete problem concerning the convergence of trigonometric series had turned into something far more profound. Further investigation led Cantor to a new theory of the infinite and the mathematical community stood with a mixture of bewilderment and disbelief before an unfamiliar universe. This theory was refined and developed, and continues to this day as axiomatic set theory. The theory of sets is a body of work that I feel should be known to a much wider audience than the mathematicians and mathematical philosophers who hold it in such high esteem.

This book is about the concepts which lie at the logical foundations of mathematics, including the rigorous notions of infinity born in the ground-breaking work of Cantor. It has been my intention to make the heroes of the book the ideas themselves rather than the multitude of mathematicians who developed them, so the treatment is deliberately sparse on biographical information, and on lengthy technical arguments. The aim is to convey the big ideas, to advertise the theory, leaving the enthused to seek further details in what is now a vast literature. This omission of technicalities is not to be interpreted as an act of laziness – in fact the effort required to hold back from giving rigorous justification for some of the items discussed herein was significant, and I have often buckled under the pressure, giving an outline of a proof or providing some technical details in the remarks at the end of each section. There are many books on the subject which flesh out these logical arguments at full length, some of which are listed in the bibliography. If an amateur reader is drawn to at least one of these as a result of reading this book then I would consider the effort of writing it worthwhile.

I cannot pretend that it is possible to write a truly 'popular' account of set theory. The subject needs a certain amount of mathematical maturity to appreciate. Added to this is the difficulty that many of the conceptually challenging

[1] 'Über das Unendliche', *Mathematische Annalen*, **95**, 161–90, (1926).

parts of the subject appear at the very beginning – the student is forced to detach himself from a comfortable but naive position – and there is little that can be done about this. A technical book on the same subject can afford to start tersely with a list of the symbols to be used together with the axioms, assuming that the motivation is already known to the reader. It can be argued that this is why axiomatic set theory is not as widely known as some other branches of mathematics. It is necessary to have amassed some experience with mathematics in order to get a firm grasp of what the abstract theory is talking about – set theory would not have been born at all without the force of centuries of mathematical research behind it, for this body of work informs the choice of axioms and the choice of basic definitions that need to be made. For this reason the introductory chapter is significantly longer than any other.

Mathematics is a gargantuan subject, bigger than most are aware. In his *Letters to a Young Mathematician*[1] Ian Stewart estimates, based on reasonable assumptions drawn from the output of the *Mathematical Reviews*, that approximately one million pages of new mathematics is published every year. One mathematician could easily spend a thousand lifetimes navigating the subject but still feel that the surface had barely been scratched; each proof or insight leads to a legion of new problems and areas to explore. Yet, despite its size, there are few regions in the expanse of mathematics that could accurately be described as remote. Two apparently different branches of the subject are never more than a few theorems away from one another, indeed some of the most interesting parts of mathematics are precisely those that forge such deep connections between seemingly distinct fields. It is a direct result of this combination of dense connectivity and huge size that in any account of the subject the temptation to wander into adjacent territory is enormous. In order to keep the book at a reasonable length I have had to be particularly disciplined, at times brutal, in controlling digressions.

Although the general public is, for the most part, rather shy of mathematics (and the sciences in general) and, more worryingly, unaware of the *existence* of mathematics beyond calculus, or even arithmetic, I have been pleasantly surprised at the genuine interest that some non-mathematicians have shown in the subject. In fact, without the encouragement of these individuals this book would most likely never have been written. Mathematics is a huge, endlessly fascinating and deeply creative subject which is alive and very well indeed; it grows significantly by the day. It is not very well advertised in the media, but it nevertheless seems to have captured the imagination of many amateurs.

Set theory has a dual role in mathematics. It can be viewed as merely another branch of the subject, just as any other, with its own peculiarities and diversions, yet it has a second role, that of a foundation of mathematics itself. This foundational role makes set theory a particularly ripe source for philosophical discussion. Each of the notions we encounter in mathematics can be modelled by particular sets or classes of objects, and consequently each part of mathematics may be 'embedded' within set theory; it is taken for granted

[1] Stewart [**202**].

that the problems of mathematics can be so expressed in set theoretic terms. We spend much of the introduction describing various classical structures, indicating how they may be translated into the new rigorous language.

There is great power in the set theoretic description of the mathematical universe. As we shall see in detail, one of Cantor's questions was, naively: '*how many points are there in a line?*'. This had little more than vague philosophical meaning before the arrival of set theory, but with set theory the question can be made precise. The punch-line in this case is that the standard theory of sets – a theory which is powerful enough to sensibly formulate the question in the first place – is not strong enough to give a definite answer, it only gives an infinite number of possible answers, any one of which can be adopted without causing contradiction. The solution remains undecidable; the theory does not know the size of the beast it has created! This flexibility in the solution is arguably far more intriguing than any definite answer would have been – it transfers our focus from a dull fixed single universe of truths to a rich multiverse, one consistent universe for each alternative answer.

Cantor's research was not initiated by lofty thoughts on the nature of the infinite. He was led to the theory through work on far more sober problems. Such humble origins are typical of grand discoveries; successful abstract theories are rarely invented just for the sake of it, there is usually a very concrete and, dare I say it, practical problem lurking at the theory's genesis. In turn, solutions to practical problems often come as corollaries of highly abstract bodies of work studied just for the love of knowledge. This lesson from history increasingly falls on deaf ears when it comes to the funding and practice of science; far-reaching research with no *immediate* practical use is always in danger of being overlooked in favour of the sort of uninspired (and uninspiring) drudgery which is worked out today only to be used tomorrow. We will meet some of the original motivating problems that helped to ignite the theory of the infinite.

Many people, when they first encounter the proof of Cantor's Theorem, which implies an infinitude of sizes of infinity, or the special case that the set of real numbers is not equipollent with the set of natural numbers, probably share a sense of injustice at being kept in the dark about such a remarkable result. This is a fairly simple proposition with an easy proof, but it is nevertheless extraordinary. *Why wasn't I told this earlier?*, *Why doesn't everyone know about this?*, might be typical reactions. That said, there is no point in introducing set theory to anyone before they have acquired an intuitive grasp of some basic mathematical ideas. I have assumed of the reader a confidence in dealing with 'abstract symbolism', for want of a better term. An important point which needs to be emphasized early on is that infinite objects generally do not exhibit the familiar combinatorial behaviour of finite objects, and nor should we have any reason to expect them to.

The language of set theory has been widely adopted; almost every branch of mathematics is written, with varying degrees of precision, in set theoretic terms. The main notions of set theory, including cardinal and ordinal numbers and the various incarnations of transfinite induction and recursion, are familiar landmarks in the landscape of mathematics.

Although some mathematicians with research interests in set theory will undoubtedly peek at the book out of curiosity, they are far from the intended audience and I'm afraid they will almost certainly find that their favourite idea, to which they have devoted a life's work, is condensed to a short sentence, or is most likely not here at all. An homuncular researcher has been perched on my shoulder throughout the writing, poking my ear with a little pitchfork whenever I have left an avenue unexplored or truncated an idea. I can only apologize if the reader feels that I have unduly neglected something.

The idea to write this book was partly triggered by a casual rereading, some time in 2004 or 2005, of a dog-eared second-hand copy of Takeuti and Zaring's *Introduction to Axiomatic Set Theory.*[1] Takeuti and Zaring's book is based on lectures delivered by Takeuti in the 1960s, not so long after Cohen's invention of forcing (one of the final subjects of this book). It is written largely in logical symbols, the occasional English sentence or paragraph providing some direction. I wondered if it was possible to write an accessible account of this opaque-looking material, and related matter, for a wider audience who did not want to become too bogged down in the symbol-crunching details. Other books have admittedly covered similar ground but none seem to discuss all of the topics I wanted to include. After an inexcusably long period of procrastination I decided that the project might be worthwhile and set to work on what turned out to be a surprisingly challenging task. Much later, in fact several years later than intended, after repeatedly retrieving abandoned copies of the manuscript out of oblivion, and after indulging in countless side-projects and distractions, a book has emerged – nothing like the book I had imagined in the early days, but nonetheless something approximating it.

I'd like to take this opportunity to thank Roger Astley at Cambridge University Press, who first showed an interest in the book, and also Clare Dennison, Charlotte Thomas and the rest of the production team for their expert handling of the project. Frances Nex copy-edited the manuscript with painstaking precision, and her suggestions greatly improved the text. If any errors remain, they are a result of my subsequent meddling. I am also most grateful to the staff of The British Library, who never failed to hunt down even the most obscure reference that I could throw at them.

Remarks

1. Each section of the book is followed by a remark or three. This gives me an opportunity to sketch any technical details, comment on peripheral topics and digress a little from the main theme without undue disruption.

2. It must be very surprising to the layman to learn that different sizes of infinity were discovered in the late nineteenth century, as if they were glimpsed through a telescope or found growing in a forgotten Petri dish. Our task is to

[1] Takeuti and Zaring [**207**].

describe exactly what it was that Cantor found and along the way perhaps indirectly clarify what mathematicians mean when they say something exists, and indeed clarify what the objects of mathematics are. Broadly speaking the approach taken by most mathematicians who are not constrained by the peculiar demands of physics or computer science is to equate coexistence with coconsistency. Mathematics is then viewed as a free and liberating game of the imagination restricted only by the need for logical consistency. In fact it is generally only philosophers of mathematics who worry about the ontology of mathematical objects – mathematicians are, quite rightly, primarily interested in producing new mathematics.

3. I have already mentioned that this book can hardly be accurately classified as a popular exposition of the subject. There is a tendency in some popular science writing to focus on the 'human story', or to create one if one cannot be found, discarding the scientific details in favour of fairly shallow journalistic accounts of the private lives of the people involved. When done well, this can certainly be diverting, if not educational, but it seems to be used as an excuse to avoid talking about the hard (and therefore interesting) stuff. If you finish the book or article thinking that the lives of the scientists and mathematicians are more interesting than the science and mathematics they discovered, then something is clearly amiss, the book can be argued to have missed its target. On the other hand, to write a serious biography is a very demanding task involving time consuming historical research. To have included such material over and above the mathematics would, I feel, have obscured the content, so my focus is entirely on the subject matter. By way of balance I have added some biographical works to the bibliography.

4. One question which is guaranteed to make a pure mathematician's blood boil is '*what are the applications?*', or the closely related '*what is it for?*'. In a more tired moment, caught off guard, there is a temptation to reply by quoting Benjamin Franklin, or was it Michael Faraday, who (probably never) said '*what use is a newborn baby?*'. The implication here is that the pure mathematics of today is the physics of tomorrow, which in turn is the everyday technology of the next century. Although history provides enough examples to assert that this is generally a fairly accurate portrait of the relationship between mathematics and the so-called real world, it is absolutely *not* the reason why most pure mathematicians do mathematics. Mathematicians are not a species of altruists hoping to improve the quality of life of their great-great grandchildren. The reason for doing pure mathematics is simply that it is paralyzingly fascinating and seems to mine deeply into the difficult questions of the Universe. That it happens to have useful spin-offs somewhere in the future is beside the point. Analyze the motivation behind the question and flinch at it. It is very difficult to predict which parts of current mathematics will find a use in tomorrow's physics, but one can confidently formulate the following two laws of mathematical utility: (i) as soon as a piece of pure mathematics is described as useless, it is destined to become extremely useful in some branch of applied mathematics or physics;

and (ii) as soon as a piece of mathematics is hailed by its creator as being ground-breaking and full of potential applications, it is destined to vanish without a trace.

5. One of the dangers of learning about Cantor's work (or any piece of pure mathematics, for that matter) without an adequate mathematical and logical background, and without the philosophical maturity that comes with it, is that there is a tendency to take the defined terms at face value and in turn react aggressively against all that is counterintuitive. That 'counterintuitive' does not mean 'false' is one of pure mathematics' prize exhibits, but it takes some effort to reach these results with clarity. We have also discovered that 'seemingly obvious' does not mean 'true' in mathematics. This phenomenon is not confined to mathematics: the physical world is also wildly counterintuitive at its extremes.

6. Mathematics was practiced for millennia without set theory. Many mathematicians continue to produce beautiful results without knowing any set theoretic axioms, and perhaps only having a vague acquaintance with the results of axiomatic set theory. Despite it acting as a foundation of mathematics, one can do mathematics without knowing axiomatic set theory. The point of set theory is not to provide a new way of finding mathematical results. Producing mathematics is, as it always has been, a matter of hard creative effort. Visual and conceptual reasoning play a critical role.

Synopsis

Proof is the idol before whom the pure mathematician tortures himself.

– Sir Arthur Eddington[1]

The vast majority of all humanly describable logical truths are, if presented without adequate preparation, either counterintuitive or beyond intuitive judgement. This is not a feature of an inherently peculiar Universe but merely a reminder of our limited cognitive ability; we have a thin grasp of large abstract structures. Fortunately we can gain access to the far reaches of such alien territory by building long strings of logical inferences, developing a new intuition as we do so. A proof of a theorem describes one such path through the darkness. Another important aspect of mathematics more closely resembles the empirical sciences, where features of a mathematical landscape are revealed experimentally, through the design of algorithms and meta-algorithms. In this book I assemble a miniature collage of a part of mathematics; an initial fragment of a huge body of work known as axiomatic set theory. The ambition of the book is a humble one – my intention is simply to present a snapshot of some of the basic themes and ideas of the theory.

Despite the impression given by the impatient practices of the media, it is not possible to faithfully condense into one convenient soundbite the details of any significant idea. One cannot hope to explain the rules of chess in six syllables, and it would be equally absurd to expect a short accessible account of set theory to be anything more than a fleeting glimpse of the whole. Perhaps the answer is not to attempt such a thing at all. My experiment was to see if something brief, coherent, yet still accurate, could be forged out of a difficult literature. I do not know if I have been successful, but I don't want the book to be an apology for omissions.

Writing prose about mathematics is notoriously difficult; the subject simply doesn't lend itself to a wordy treatment. At its worst an expository article can be a cold and off-putting list of facts punctuated by indigestibly dense passages of definitions. Technical texts do not provoke such *ennui* because the reader is an active participant in the theory, investing a lot of time on each assertion, carefully exploring adjacent ideas in an absorbing stream of consciousness journey, so that any list-like character of the original text is disguised by the slow

[1] Eddington [**60**]. Chapter 15.

motion of digestion and digression. The message is that passive reading benefits nobody. Many technical mathematical texts are designed to be read more than once, or are to be dipped into, rarely being read from cover to cover (at least not in the order of pagination). I have, naturally, written the text so that it can be read in sequence, but I will be neither surprised nor insulted if the reader skips back and forth between chapters and there may even be some benefit in doing this.

In any foundational investigation one inevitably feels an overpowering sense that 'everything I thought I knew is wrong', which is always a healthy development, even if it is a little disorientating at first. To those who never decamp from the foothills of such work, however, the proceedings can seem like a pointless exercise in pedantry, or even a reinvention of the wheel in obscure terms. I have tried to avoid giving this false impression, although there are some necessary preliminaries concerning familiar number systems which need to be covered before moving ahead to less intuitive matters.

Mathematicians who dip their toes into foundational philosophical matters often become the victims of a form of digressional paralysis, which may well be a pleasant state of mind for philosophers but is a bit of a hindrance for the mathematician who would rather produce concrete results than chase rainbows. Like a participant in Zeno's race, not being able to start the journey, the traveller must first argue about the sense of 'existence' and then argue about the sense of 'sense'. There are many statements herein which could easily be dismantled and turned into a full-blown dialogue between philosophers, logicians, mathematicians and physicists in a devil's staircase of counteropinions. I have tried, for the sake of brevity and sanity, not to indulge too frequently in this microscopic analysis of each and every assertion. Stones left unturned in good quantity make fine fodder for debate in any case.

Although the ideas discussed in this book are now well-established in mainstream mathematics, in each generation since Cantor there have always been a few individuals who have argued, for various philosophical reasons, that mathematics should confine itself to finite sets. These we shall call Finitists. Despite my occasional sympathies with some of their arguments, I cannot claim to be a Finitist. I'm certainly a 'Physical Finitist' to some extent, by which I mean that I am doubtful of the existence of infinite physical aggregates, however I do not extend this material limitation to the Platonic constructions of mathematics (the set of natural numbers, the set of real numbers, the class of all groups, etc.), which seem perfectly sound as *ideas* and which are well-captured in the formalized language of set theory. However, I cannot pretend to be a strong Platonist either (although my feelings often waver), so my objection is not based on a quasi-mystical belief that there is a mathematical infinity 'out there' somewhere. In practice I find it most convenient to gravitate towards a type of Formalism, and insofar as this paints a picture of mathematics as a kind of elaborate game[1] played with a finite collection of symbols, this makes me a Finitist in a certain Hilbertian sense – but I have no difficulty in *imag-*

[1] If it is a game, then Nature seems to be playing by the same rules!

ining all manner of infinite sets, and it is from such imaginings that a lot of interesting and very useful mathematics springs, not from staring at or manipulating strings of logical symbols.[1] I am not alone in being so fickle as to my philosophical leanings; ask a logician and a physicist to comment on the quote attributed to Bertrand Russell '*any universe of objects...*' which opens Section 11.1 of Chapter 11 and witness the slow dynamics of shifting opinions and perspectives as the argument unfolds.

The history of mathematical ideas is often very difficult to untangle, and because ideas evolve gradually over a long period of time it is impossible to draw an exact boundary between a given theory and its offspring. It is consequently a painful task to credit some developments to a small number of authors. On occasion I have had to resort to the less than satisfactory device of mentioning only a few of the prominent names. On those occasions when I label an idea with a date it is because that is the year that the idea, possibly in its final form, was published. Behind each date is a thick volume of information and behind every bold statement is a host of qualifications which, if included in their entirety, would render the present book an unpalatable paper brick. In an ideal world it would be possible to attach a meta-commentary to the text, extending the dimensions of the book. A skillfully constructed website might be able to achieve this, but then there would be a lack of cohesion and permanence, and a slimmer chance that all will be read.

I have indulged in 'strategic repetition', a luxury which is uniquely available in writing at this less than technical level. I have done this both to reinforce ideas and to view the same objects from different perspectives. I hope the reader will forgive me for telling them the same thing, albeit from a different point of view, on more than one occasion. For example, I point out several times the equivalence of the Well-Ordering Theorem and the Axiom of Choice, and, under the influence of the Axiom of Choice, the equivalence of different ways of looking at cardinal and ordinal numbers. I also repeat the fact of the absence of atoms in ZF. I wish to interpret these repetitions – in particular the material which is recast in more precise terms from its intuitive base in the introduction – simply as motifs revisited against a clearer background.

There are one or two occasions when I commit the logical *faux pas* of mentioning a concept before it is properly defined. When this happens, the reader should simply log the intuitive idea as a proto-definition, ready for future rigorous clarification. The logically sound alternative turns a slightly turbulent prose into a tense pile-up of preliminaries awaiting resolution, leaving the reader to agonize over where it is all leading. There is also a sizeable conceptual distance between some of the simple ideas covered in the earlier part of the introduction and the more technical matter towards the end of the book.

[1]Formalism is sometimes unfairly painted as merely a naive way of avoiding philosophical discussion (more precisely as a tactic used to dodge ontological commitment). Even more unfairly it is presented as being sterile and, perhaps worst of all, as being associated in some way with the notion that mathematics is invented and not discovered. Some fashions come and go. The ever growing literature on competing philosophies of mathematics could fill a library and plausible arguments can be found for almost any imaginable position.

In the first section we meet the core idea of cardinality together with associated terminology and notation. Once sufficient motivation has been generated the formal constructions of the classical number systems are described. It should already be clear at this stage that the reader is expected to have an intuitive familiarity with these systems before viewing them in these formal terms. The point of the exercise is to look at these structures anew and to attempt to put them all on the same logical footing. Once the hurdle of constructing the natural numbers has been cleared the rest is relatively smooth going thanks to some simple algebraic machinery. Although the introduction only covers the basics we can afford to give a fairly detailed exposition. The passage from the natural numbers to the complex numbers is described in its entirety, while the developments of later chapters necessarily become progressively more fragmented. The main goal of the first part of the introduction is to describe the rigorous construction of the real numbers (the complex numbers being a simple extension). This gives us an opportunity to view the continuum from various different angles – it appears as the class of all Dedekind cuts; certain Cauchy sequence equivalence classes; and synthetically as a complete ordered field. Some interesting bijections are presented and, most importantly, via Cantor's well-known diagonal argument, the strict cardinal difference between the set of natural numbers and the set of real numbers is demonstrated. The algebraic numbers are the last of the classical number types to be discussed.

Having introduced the cardinal infinite, the limiting processes associated with calculus are described; this material is a quick sketch of the theory that students usually meet early on in their study of analysis. The operations of calculus are presented in non-mysterious terms, removing the need for undefined 'infinitesimals'. In terms of condensing information into almost nothing the latter in particular imitates a whale being forced through the eye of a needle. The history of calculus is lengthy and deserves a multi-volume treatment. This is equally true, as I stress (squirming at my own self censorship), of the later treatment of complex numbers.

The 'order type' beginnings of ordinal numbers are detailed and Cantor's approach to the ordinal building process is described. This is material which is to be viewed again from various different perspectives (all equivalent assuming the Axiom of Choice) later in the book. The ordinal notion is revealed to have its roots in analysis via iterative processes which fail to stabilize after 'ω' many iterations. The Russell–Whitehead realization of cardinal numbers is repeated and the cardinal arithmetic operations are described – again these are to be revisited later.

I have indulged in a little fantasy in the section on the Finite Universe for the purpose of defending a finitistic view of the physical Universe (not in itself a controversial viewpoint but a useful contrast to the infinite implied in purely mathematical structures). Following this a brief tribute to Zeno's paradoxes is given.

The Axiom of Choice, the Continuum Hypothesis and its generalization are central to the text. In the introduction we meet them briefly, setting the scene for future discussion. The introduction ends with an impression of what is to come.

Chapter 2 through to the end might be regarded as one long endnote, describing in more formal terms what is needed to place the ideas of the introduction on a sound logical foundation. What do we assume at the outset, what are the primitive terms and sentences of the language? What are the logical axioms and rules of inference? At this point we adopt Zermelo–Fraenkel set theory (ZF) as our principal theory and all future statements are to be interpreted within this biased framework. Here we also meet the first non-logical axiom of ZF, Extensionality.

Russell's paradox must be regarded as a major catalyst in the development of modern set theory, although it is of course not the only one. The origin of the paradox is spelled out and some of its relatives are described. We try to identify what needs to be avoided in order to resolve the paradox and mention some strategies. Classes are introduced as defined terms and the formal development of this theory is outlined. Set hierarchies are discussed in general and we get a first glimpse of the von Neumann set hierarchy.

Further non-logical axioms appear: Pairing, Unions, Powers, Replacement and Regularity. We return to the idea of order in a closer look at cartesian products and relations, foundational relations, and the all important notion of well-ordering which underpins ordinal numbers. Ordinal numbers reappear in a new disguise and we meet the last of the non-logical axioms of ZF, the Axiom of Infinity. A Formalist approach is adopted to avoid interpretative controversy. Having opened the gates, via the last axiom, to the huge hierarchy $\mathcal{O}n$ of all ordinal numbers we introduce the powerful methods of transfinite induction and transfinite recursion. The arithmetic of ordinals is described and we return to the von Neumann hierarchy to look at the elementary notion of rank. Having reconstructed ordinal numbers in purely set theoretic terms we revisit cardinal numbers, again stressing different perspectives and exhibiting different definitions of 'finite'. We also look at the upper cardinal boundaries imposed by the axioms of ZF and the Axiom of Choice and examine new axioms needed to break through these inaccessible heights.

We are reacquainted with the ZF-undecidable Axiom of Choice and the Continuum Hypothesis. Some well-known equivalents of the former are described and the Banach–Tarski paradox makes an appearance, a result which is sometimes (misguidedly) given as a reason to reject the Axiom of Choice. Alternatives to the Axiom of Choice are discussed.

The theory of models is introduced, a subject which removes a lot of mystery from set theory, giving us something tangible to work with when proving metamathematical statements and, in particular, allowing us to gain access to independence results. The final chapter, necessarily sketchy, attempts to describe how independence results are proved by modifying existing models. The focus is on the Axiom of Constructibility. First we meet Gödel's constructible universe, a model of ZF in which the Axiom of Choice and the Continuum Hypothesis both hold. Finally we indicate some of the classical ideas that were used to generate a model in which Constructibility fails. The latter material is intended to be little more than a 'six syllable rules of chess' account of the topic, although more detail is given than most popular texts would dare. The book must end here, for further details would demand a more thorough technical grounding.

I have included three appendices. Appendix A briefly describes Peano Arithmetic, the first-order formalization of the arithmetic of natural numbers. Appendix B collects together the axioms of ZF which are discussed in more detail in the text. Appendix C is my attempt to give a postage stamp sized account of Gödel's Incompleteness Theorems. This should be regarded as 'impressionistic' at best and for the purposes of the main text of the book it suffices to know that in Peano Arithmetic and Zermelo–Fraenkel set theory (and in similar theories) there exist arithmetical statements which cannot be proved or disproved, and that such theories cannot prove their own consistency.

If more details on set theory are sought then it is time to mine the bibliography for items of interest; I have provided a little summary of the contents of each listed item. I urge the reader to at least browse through it to get an impression of the size of the subject and to see what little I have been able to cover here. The bibliography is broad, ranging from popular texts readable by the layman to fiendishly technical monographs demanding many years of intense study. Due to the small readership a lot of these items are now regrettably out of print, so the collector must be prepared to hunt for an affordable second hand copy or head for a good library. I must add that there is a torrent of current research on the subject which is not even hinted at either in the text or in the bibliography.

Among many other texts, I should record the influence of Michael Potter's *Set Theory and its Philosophy*,[1] which I discovered only part way through the first draft of this book, and, to a lesser but still significant extent, Mary Tiles' *The Philosophy of Set Theory*.[2] Takeuti and Zaring's *Introduction to Axiomatic Set Theory*[3] was, as mentioned in the preface, the original motivating text, and its presence, as well as that of Cohen's pioneering exposition of his method,[4] can still be felt in some of the more technical sections in the latter half of the book. Raymond Smullyan's *Gödel's Incompleteness Theorems*[5] provided a useful template for Appendix C. Even if the content of the many texts that I consulted when writing this book is not reflected herein, the sole fact of their existence provided some inspiration to continue writing.

Just one last word is in order concerning the quotations at the beginning of each section. In writing this book I have discovered something I have long suspected to be true: most famous quotations are either incorrect, misattributed or fictitious. I have been careful to give fuller quotes where possible, setting them in their proper context, and in most cases I have given the primary source of the quote.

[1]Potter [**170**].
[2]Tiles [**212**].
[3]Takeuti and Zaring [**207**].
[4]Cohen [**34**].
[5]Smullyan [**198**].

Remarks

1. I think I ought to clarify the simple distinction between 'counterintuitive' and 'beyond intuitive' as an addendum to my opening paragraph. Both rely on a plastic intuitive base of results entirely dependent on the experience of the individual. A true result is counterintuitive to an individual if it seems to contradict his intuitive grounding. For example, the Banach–Tarski paradox, which we will meet later (see Subsection 9.1.5), is counterintuitive to anyone who thinks of geometry in purely physical terms. Through careful study or an appropriate change of perspective a counterintuitive result can in time become intuitive, the intuitive ground now suitably remoulded. A result or conjecture that is beyond intuitive judgment is one that really doesn't seem to relate to the intuitive base of results at all, so it is difficult to make any call as to whether it might be true or not. To find examples of such statements go to a conference or open a book in a branch of mathematics you know nothing about. Find a conjecture that the sub-community agrees on: 'it seems likely that every Hausdorff quasi-stratum is a Galois regular connector' and there is often no hint as to where this conjecture might have come from, even after you learn what Hausdorff quasi-strata and Galois regular connectors are.[1] Spend a few years studying it and you start to get an idea of why the conjecture was made, and eventually it might even become intuitively obvious to you, and lead to further conjectures. Of course there are statements which are beyond anyone's grasp, and may remain so forever; since we can only ever understand finitely many things, and since in any symbolic logic one can generate a potentially infinite number of implications, most statements will fall into the latter inaccessible class.

2. Studying technical mathematics often seems to involve more writing than reading. One sentence can trigger a good few pages of notes (drawings, diagrams, misproofs, proofs, dead ends), first in an attempt to fully understand what the sentence says, and then a series of playful modifications of the assumptions, moving off on various related threads, extending the results in some direction.

3. Axioms in the Euclidean sense are intended to be 'self-evident truths'. Modern axioms need not be so obvious. They are chosen for other reasons, often as distillations of observed phenomena in existing theories. The emphasis is more on the consequences of the new assumptions rather than any immediate intuitive plausibility of the assumptions themselves. An axiom, as it is meant in the modern sense, is just an assumption whose consequences one can explore.

[1] To put the reader at ease, these two terms are (as far as I know) my own inventions.

4. Mentioning a concept before giving its formal definition is not quite the criminal act that some would pretend it to be. In his sometimes controversial article *The Pernicious Influence of Mathematics Upon Philosophy*[1] Gian-Carlo Rota stresses the distinction between mathematical and philosophical enquiry, mathematics starting with a definition and philosophy ending with one. Even within mathematics definitions and theorems are mutually influential.

5. I should stress the importance of having an intuitive understanding of the common number systems before looking at any formal definitions. As just mentioned, the formal definitions we make are made only after thoroughly examining the underlying intuitive ideas, and they are not made with the intention of explaining what a number really is, whatever that might mean.

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[1] See Rota [**178**].