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Calendar Basics

A learned man once asked me regarding the eras used by different nations, and regarding the difference of their roots, that is, the epochs where they begin, and of their branches, that is, the months and years, on which they are based; further regarding the causes which led to such difference, and the famous festivals and commemoration-days for certain times and events, and regarding whatever else one nation practices differently from another. He urged me to give an explanation, the clearest possible, of all this, so as to be easily intelligible to the mind of the reader, and to free him from the necessity of wading through widely scattered books, and of consulting their authors. Now I was quite aware that this was a task difficult to handle, an object not easily to be attained or managed by anyone, who wants to treat it as a matter of logical sequence, regarding which the mind of the student is not agitated by doubt.

Abū-Raiḥān Muḥammad ibn ’Aḥmad al-Bīrūnī:
Al-Āthār al-Bāqiyah ‘an al-Qurūn al-Khāliyah (1000)

Calendrical calculations are ubiquitous. Banks need to calculate interest on a daily basis. Corporations issue paychecks on weekly, biweekly, or monthly schedules. Bills and statements must be generated periodically. Computer operating systems need to switch to and from daylight saving time. Dates of secular and religious holidays must be computed for consideration in planning events. Most of these calculations are not difficult because the rules of our civil calendar (the Gregorian calendar) are straightforward.

Complications begin when we need to know the day of the week on which a given date falls or when various religious holidays based on other calendars occur. These complications lead to difficult programming tasks—not often difficult in an algorithmic sense but difficult because it can be extremely tedious to delve into, for example, the complexities of the Hebrew calendar and its relation to the civil calendar.

The purpose of this book is to present, in a unified, completely algorithmic form, a description of over three dozen calendars and how they relate to one another. Among them are included the present civil calendar (Gregorian); the recent ISO commercial calendar; the old civil calendar (Julian); the ancient Egyptian calendar and its Armenian equivalent; the Coptic and the virtually identical Ethiopic calendars; the Akan (African) calendar, the Islamic (Muslim) calendar (the arithmetical version, one based on calculated lunar observability, and a

Saudi Arabian variant); the modern Persian calendar (both astronomical and arithmetic forms); the Bahá'í calendar, both arithmetic and astronomical forms; the Hebrew (Jewish) calendar, both its present arithmetical form and a speculative observational form; the three Mayan calendars and two virtually identical Aztec calendars; the Pawukon calendar from Bali; the French Revolutionary calendar (both astronomical and arithmetic forms); the Chinese calendar and the virtually identical Japanese, Korean, and Vietnamese calendars; both the old (mean) and new (true) Hindu (Indian) solar and lunisolar calendars; and the Tibetan calendar. Information that is sufficiently detailed to allow computer implementation is difficult to find for most of these calendars because the published material is often inaccessible, ecclesiastically oriented, incomplete, inaccurate, based on extensive tables, overburdened with extraneous material, focused on shortcuts for hand calculation to avoid complicated arithmetic or to check results, or difficult to find in English. Most existing computer programs are proprietary, incomplete, or inaccurate.

The need for such a secular, widely available presentation was made clear to us when we (primarily E.M.R., with contributions by N.D.), in implementing a calendar/diary feature for GNU Emacs [44], found difficulty in gathering and interpreting appropriate source materials that describe the interrelationships among the various calendars and the determination of the dates of holidays. Some of the calendars (Chinese, Japanese, Korean, Vietnamese, Hindu, and Tibetan) had never had full algorithmic descriptions published in English.

The calendar algorithms in this book are presented as mathematical function definitions in standard mathematical format. Appendix A gives the types (ranges and domains) of all functions and constants we use; Appendix B is a cross reference list that gives all dependencies among the functions and constants. In Appendix C we tabulate results of the calendar calculations for 33 sample dates and 44 holidays; this will aid those who develop their own implementations of our calendar functions. To ensure correctness, all calendar functions were automatically typeset[1] directly from the working Common Lisp [46] functions given in Appendix D.[2]

We chose mathematical notation as the vehicle for presentation because of its universality and easy convertibility to any programming language. We have endeavored to simplify the calculations as much as possible without obscuring the intuition. Many of the algorithms we provide are considerably more concise than previously published ones; this is particularly true of the arithmetic Persian, Hebrew, and old Hindu calendars.

We chose Lisp as the vehicle for implementation because it encourages functional programming and has a trivial syntax, nearly self-evident semantics, historical durability, and wide distribution; moreover, Lisp was amenable to translation into ordinary mathematical notation. Except for a few short macros, the code uses

[1] This has meant some sacrifice in the typography of the book; we hope readers sympathize with our decision.

[2] The Lisp code is available through a Cambridge University Press web site `www.cambridge.org/calendricalcalculations` under the terms of the License Agreements and Limited Warranty on page xli. Any errata are available over the World Wide Web at `www.calendarists.com`.

only a very simple, side-effect-free, subset of Lisp. We emphasize that our choice of Lisp should be considered irrelevant to most readers, whom we expect to follow the mathematical notation used in the text, not to delve into the code.

It is not the purpose of this book to give a detailed historical treatment of the material, nor, for that matter, a mathematical one; our goal is to give a logical, thorough, *computational* treatment. Thus, although we give much historical, religious, mathematical, and astronomical data to leaven the discussion, the focus of the presentation is algorithmic. Full historical and religious details as well as the mathematical and astronomical underpinnings of the calendars can be pursued in the references.

In this chapter, we describe the underlying unifying theme of all the calculations along with some useful mathematical facts. The details of specific calendars are presented in subsequent chapters. Historically, the oldest calendars that we consider are the Egyptian (more than 3000 years old) and Babylonian. The Chinese and Mayan calendars also derive from millennia-old calendars. Next are the classical (observation-based) Hebrew, the Julian (the roots of which date back to the ancient Roman empire), the Coptic and Ethiopic (third century), the current Hebrew (fourth century) and the old Hindu (fifth century), followed by the Islamic calendar (seventh century), the newer Hindu calendars (tenth century), the Persian and Tibetan calendars (eleventh century), the Gregorian modification to the Julian calendar (sixteenth century), the French Revolutionary calendar (eighteenth century), and the Bahá'í calendar (nineteenth century). Finally, the International Organization for Standardization's ISO calendar and the arithmetic Persian calendar are of twentieth-century origin.

For expository purposes, however, we present the Gregorian calendar first, in Part I, because it is the most popular calendar currently in use. Because the Julian calendar is so close in substance to the Gregorian, we present it next, followed by the very similar Coptic and Ethiopic calendars. Then we give the ISO calendar and the Icelandic calendar, which are trivial to implement and depend wholly on the Gregorian. The arithmetic Islamic calendar, which because of its simplicity is easy to implement, follows. Next, we present the Hebrew calendar, one of the more complicated and difficult calendars to implement. This is followed by a chapter on the computation of Easter, which is lunisolar like the Hebrew calendar. The ancient Hindu solar and lunisolar calendars are described next; these are simple versions of the modern Hindu solar and lunisolar calendars described in Part II. Next come the Mayan and similar Aztec calendars of historical interest, which have several unique computational aspects. These are followed by the Balinese Pawukon calendar. All the calendars described in Part I are "arithmetical" in that they operate by straightforward integer-based rules. We conclude Part I with a chapter describing the generic arithmetic calendar schemata that apply to many calendars in this part.

In Part II we present calendars that are controlled by irregular astronomical events (or close approximations to them), although these calendars may have an arithmetical component as well. Because the calendars in Part II require some understanding of astronomical events such as solstices, equinoxes, and lunar phases, we begin Part II with a chapter introducing the topics and algorithms that will be needed. We then give the modern Persian calendar in its astronomical and arithmetic forms followed by the Bahá'í calendar, also in two versions: the former Western (arithmetic) version, which depends wholly on the Gregorian, and the new astronomical version. Next we describe the original (astronomical) and modified (arithmetic) forms of the French Revolutionary calendar. All these calendars are computationally simple, provided that certain astronomical values are available. Next we describe some astronomical calendars based on the moon: the Babylonian calendar, a proposed astronomical calculation of Easter, the observational Islamic calendar, and the classical Hebrew calendar. We continue with the Chinese lunisolar calendar and its Japanese, Korean, and Vietnamese versions. We then describe the modern Hindu calendars, which are by far the most complicated of the calendars in this book. We conclude with the Tibetan calendar.

We also provide algorithms for computing holidays based on most of the calendars. In this regard we take the ethnocentric view that our task is to compute the dates of holidays in a given *Gregorian year*; there is clearly little difficulty in finding the dates of, say, Islamic New Year in a given Islamic year! In general we have tried to mention significant holidays on the calendars we cover but have not attempted to be exhaustive and to include all variations. The interested reader can find extensive holiday definitions in [22], [23], and [24].

The selection of calendars we present was chosen with two purposes: to include all common modern calendars and to cover all calendrical techniques. We do not give all variants of the calendars we discuss, but we have given enough details to make any calendar easy to implement.

1.1 Calendar Units and Taxonomy

Teach us to number our days, that we may attain a wise heart.
Psalms 90:12

The sun moves from east to west, and night follows day with predictable regularity. This apparent motion of the sun as viewed by an earthbound observer provided the earliest time-keeping standard for humankind. The day is, accordingly, the basic unit of time underlying all calendars, but various calendars use different conventions to structure days into larger units: weeks, months, years, and cycles of years. Different calendars also begin their day at different times: the French Revolutionary day, for example, begins at true (apparent) midnight; the Islamic, Bahá'í, and Hebrew days begin at sunset; the Hindu day begins at sunrise. The various definitions of *day* are surveyed in Section 14.3.

The purpose of a calendar is to give a name to each day. The mathematically simplest naming convention would be to assign an integer to each day; fixing day 1 would determine the whole calendar. The Babylonians had such a day count (in base 60). Such *diurnal* calendars are used by astronomers (see Section 14.3) and by calendarists (see, for example, Section 10.1); we use a day numbering in this book as an intermediate device for converting from one calendar to another (see the following section). Day-numbering schemes can be complicated by using a mixed-radix system [28] in which the day number is given as a sequence of numbers or names (see Section 1.10). The Mayans, for example, utilized such a method (see Section 11.1).

Calendar day names are generally distinct, but this is not always the case. For example, the day of the week is a calendar, in a trivial sense, with infinitely many days having the same day name (see Section 1.12). A 7-day week is almost universal today. In many cultures, the days of the week were named after the seven “wandering stars” (or after the gods associated with those heavenly bodies), namely, the sun, the moon, and the five planets visible to the naked eye—Mercury, Venus, Mars, Jupiter, and Saturn. In some languages—Arabic, Lithuanian, Portuguese, Ukrainian, and Hebrew are examples—some or all of the days of the week are numbered, not named. In the Armenian calendar, for example, the days of the week are named as follows [22, vol. 3, p. 70]:

Sunday	Kiraki (or Miashabathi)
Monday	Erkoushabathi
Tuesday	Erekhshabathi
Wednesday	Chorekhshabathi
Thursday	Hingshabathi
Friday	Urbath (or Vetsshabathi)
Saturday	Shabath

“Shabath” means “day of rest” (from the Hebrew), “Miashabathi” means the first day following the day of rest, “Erkoushabathi” is the second day following the day of rest, and so on. The Armenian Christian church later renamed “Vetsshabathi” as “Urbath,” meaning “to get ready for the day of rest.” Subsequently, they declared the first day of the week as “Kiraki” or “the Lord’s day.”

Other cycles of days have also been used, including 4-day weeks (in the Congo), 5-day weeks (in other parts of Africa, in Bali, and in Russia in 1929), 6-day weeks (Japan), 8-day weeks (in yet other parts of Africa and in the Roman Republic), and 10-day weeks (in ancient Egypt and in France at the end of the eighteenth century; see page 282). The mathematics of cycles of days are described in Section 1.12. Many calendars repeat after one or more years. In one of the Mayan calendars (see Section 11.2), and in many preliterate societies, day names are recycled every year. The Chinese calendar uses a repeating 60-name scheme for days and years, and at one time used it to name months.

An interesting variation in some calendars is the use of two or more cycles running simultaneously. For example, the Mayan tzolkin calendar (Section 11.2) combines a cycle of 13 names with a cycle of 20 numbers. The Chinese cycle of 60 names for years is actually composed of cycles of length 10 and 12 (see Section 19.4). The Balinese calendar takes this idea to an extreme; see Chapter 12. The mathematics of simultaneous cycles is described in Section 1.13.

The notions of “month” and “year,” like the day, were originally based on observations of heavenly phenomena, namely the waxing and waning of the moon, and the cycle of seasons, respectively. The lunar cycle formed the basis for the palaeolithic marking of time (see [32] and [13]), and many calendars today begin each month with the new moon, when the crescent moon first becomes visible (as in the Hebrew calendar of classical times and in the religious calendar of the Muslims today—see Sections 14.9 and 18.4); others begin the month at full moon (in northern India, for example)—see page 160. For calendars in which the month begins with the observed new moon, beginning the day at sunset is natural.

Over the course of history, many different schemes have been devised for determining the start of the year, usually based on the solar cycle.[3] Some are astronomical, beginning at the autumnal or spring equinox, or at the winter or summer solstice. Solstices are more readily observable; either one can note when the midday shadow of a gnomon is longest (at the winter solstice in the northern hemisphere) or shortest (at the summer solstice) or one can note the point in time when the sun rises or sets the farthest south during the course of the year (which is the start of winter in the northern hemisphere) or the farthest north (the start of summer). The ancient Egyptians began their year with the *heliacal rising* of Sirius—that is, on the day when the Dog Star Sirius (the brightest fixed star in the sky) can first be seen in the morning after a period during which the sun's proximity to Sirius makes the latter invisible to the naked eye. The Pleiades ("Seven Sisters") were used by the Maoris and other peoples for the same purpose. Various other natural phenomena such as harvests or the rutting seasons of certain animals have been used among North American tribes [9] to establish the onset of a new year. And not just humans use such phenomena: the lunar cycle determines life cycle events for certain corals [7], birds [43], and monkeys [30]. It has also been suggested [10] that the pink "skylight" on the crown of the head of leatherback turtles serves to allow them to determine when in late summer the lengths of day and night are equal (taking refraction into account), at which point foraging turtles turn south.

Calendars have, of necessity, an integral number of days in a month and an integral number of months in a year. However, these astronomical periods—day, month, and year—are incommensurate: their periods do not form integral multiples of one another. The lunar month is about $29\frac{1}{2}$ days long, and the solar year is about $365\frac{1}{4}$ days long (Chapter 14 has precise definitions and values). How exactly one coordinates these time periods and the accuracy with which they approximate their astronomical values is what differentiates one calendar from another.

Broadly speaking, solar calendars—including the Egyptian, Armenian, Persian, Gregorian, Julian, Coptic, Ethiopic, ISO, French Revolutionary, and Bahá'í —are based on the yearly solar cycle, whereas lunar and lunisolar calendars—such as the Islamic, Hebrew, Hindu, Tibetan, and Chinese—take the monthly lunar cycle as their basic building block. Most solar calendars are divided into months, but these months are divorced from the lunar events; they are sometimes related to the movement of the sun through the 12 signs of the zodiac, notably in the Hindu solar calendars (see Chapter 20).

Because observational methods suffer from vagaries of weather and chance, they have for the most part been supplanted by calculations. The simplest option

[3] It has been claimed that in equatorial regions, where the tropical year is not of paramount agricultural importance, arbitrary year lengths are more prevalent, such as are found in the 210-day Balinese Pawukon calendar (Chapter 12) and the 260-day Mayan divine year (Section 11.2).

is to approximate the length of the year, of the month, or of both. Originally, the Babylonian solar calendar was based on 12 months of 30 days each (see [26]), overestimating the length of the month and underestimating the year; see Figure 1.1. Such a calendar is easy to calculate, but each month begins at a slightly later lunar phase than the previous, and the seasons move forward slowly through the year. The ancient Egyptian calendar achieved greater accuracy by having 12 months of 30 days plus 5 extra days—Egyptian mythology includes a tale of how the calendar came to have these five extra days [3]. Conversions for this calendar are illustrated in Section 1.11. To achieve better correlation with the motion of the moon, one can instead alternate months of 29 and 30 days. Twelve such months, however, amount to 354 days—more than 11 days short of the solar year.

Almost every calendar in this book and virtually all other calendars incorporate a notion of "leap" year to deal with the cumulative error caused by approximating a year by an integral number of days and months.[4] Solar calendars add a day every few years to keep up with the astronomical year. The calculations are simplest when the leap years are evenly distributed and the numbers involved are small; for instance, the Julian, Coptic, and Ethiopic calendars add 1 day every 4 years. Formulas for the evenly distributed case, such as when one has a leap year every fourth or fifth year, are derived in Section 1.14. The old Hindu solar calendar (Chapter 10) follows such a pattern; the arithmetical Persian calendar almost does (see Chapter 15). The Gregorian calendar, however, uses an uneven distribution of leap years but a relatively easy-to-remember rule (see Chapter 2). The modified French Revolutionary calendar (Chapter 17) included an even more accurate but uneven rule.

Most lunar calendars incorporate the notion of a year. Purely lunar calendars may approximate the solar year with 12 lunar months (as does the Islamic), though this is about 11 days short of the astronomical year. Lunisolar calendars invariably alternate 12- and 13-month years, according either to some fixed rule (as in the Hebrew calendar) or to an astronomically determined pattern (Chinese and modern Hindu). The so-called *Metonic cycle* is based on the observation that 19 solar years contain almost exactly 235 lunar months. This correspondence, named after the Athenian astronomer Meton (who published it in 432 B.C.E.) and known much earlier to ancient Babylonian and Chinese astronomers, makes a relatively simple and accurate fixed solar/lunar calendar feasible. The $235 = 12 \times 12 + 7 \times 13$ months in the cycle are divided into 12 years of 12 months and 7 leap years of 13 months. The Metonic cycle is used in the Hebrew calendar (Chapter 8) and for the calculation of Easter (Chapter 9).

The more precise the mean year, the larger the underlying constants must be. For example the Metonic cycle is currently accurate to within 6.5 minutes a year, but other lunisolar cycles are conceivable: 3 solar years are approximately 37 lunar months with an error of 1 day per year; 8 years are approximately 99 months with an error of 5 hours per year; 11 years are approximately 136 months with

[4] See [6, pp. 677–678] for a discussion of the etymology of the term "leap."

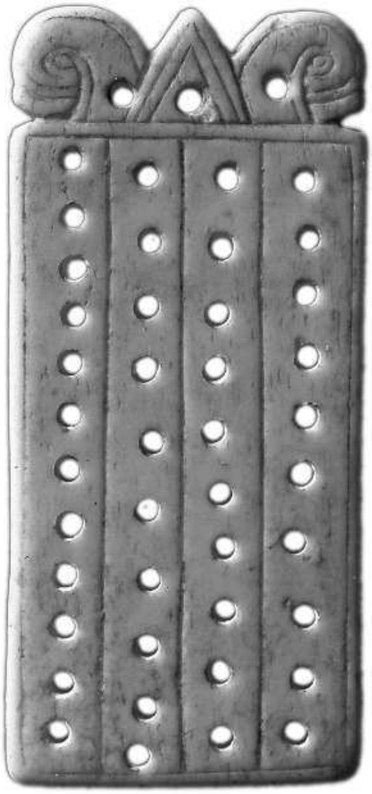

Figure 1.1 A small (6 × 2.7 cm) bone plaque found in Tel ‘Aroer, an Iron Age II (8th–6th century B.C.E.) caravan town in the Negev, Israel. It is conjectured to be a calendar counter: a peg could move daily through the 30 holes in the three right-hand columns of 10 holes each, while another peg moved monthly through the 12 holes in the first column. It could have been used either as a schematic 360-day calendar or as a lunar calendar, in which case some months would end after 29 days [14]. (Reproduced courtesy of the Hebrew Union College, Jerusalem.)

an error of 3 hours per year; and 334 years are 4131 months with an error of 7.27 seconds per year. The old Hindu calendar is even more accurate, comprising 2226389 months in a cycle of 180000 years (see Chapter 10) to which the leap-year formulas of Section 1.14 apply, and errs by less than 8 seconds per year.

The placement of leap years must make a trade-off between two conflicting requirements: small constants defining a simple leap year rule of limited accuracy versus greater accuracy at the expense of larger constants, as the examples in the last paragraph suggest. The choice of the constants is aided by taking the continued fraction (see [27]) of the desired ratio and choosing among the convergents (where to stop in evaluating the fraction). In the case of lunisolar calendars, the solar year is about 365.24244 days, while the lunar month is about 29.53059 days, so we write

$$\frac{365.24244}{29.53059} = 12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{18 + \cfrac{1}{3 + \cdots}}}}}}}.$$

By choosing further and further stopping points, we get better and better approximations to the true ratio. For example,

$$12 + \cfrac{1}{2 + \cfrac{1}{1}} = \frac{37}{3}$$

while

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}} = \frac{99}{8}$$

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}} = \frac{136}{11}$$

and

$$12 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}}} = \frac{235}{19}$$

These are the ratios of the previous paragraph. Not all approximations must come from continued fractions, however: 84 years are approximately 1039 lunar months with an error of 33 minutes per year, but this is not one of the convergents.

Continued fractions can be used to get approximations to solar calendars too. The number of days per solar year is about 365.242177, which we can write as

$$365.242177 = 365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{5 + \cdots}}}}}}$$

The convergents are 1/4 (the basis of the Julian, Coptic, and Ethiopic calendars), 7/29, 8/33 (possibly used for an ancient Persian calendar), 23/95, and 31/128 (used in our implementation of the arithmetical Persian calendar—see Chapter 15).

Table 1.1 gives for comparison the values for the mean length of the year and month as implemented by the various solar, lunar, and lunisolar calendars in this book. The true values change over time, as explained in Chapter 14.

1.2 Fixed Day Numbers

May those who calculate a fixed date . . . perish.[5]
Morris Braude: *Conscience on Trial: Three Public Religious Disputations between Christians and Jews in the Thirteenth and Fifteenth Centuries* (1952)

Over the centuries, human beings have devised an enormous variety of methods for specifying dates.[6] None are ideal computationally, however, because all have idiosyncrasies resulting from attempts to coordinate a convenient human labeling with lunar and solar phenomena.

For a computer implementation, the easiest way to reckon time is simply to count days. Fix an arbitrary starting point as day 1 and specify a date by giving a day number relative to that starting point; a single 32-bit integer allows the representation of more than 11.7 million years. Such a reckoning of time is, evidently, extremely awkward for human beings and is not in common use, except among astronomers, who use *julian day numbers* to specify dates (see Section 1.5), and calendarists, who use them to facilitate conversion among calendars—see equation

[5] This is a loose translation of a famous dictum from the Babylonian Talmud *Sanhedrin* 97b. The omitted words from Braude's translation (p. 112 of his book) are "for the coming of the Messiah." The exact Talmudic wording is "Blasted be the bones of those who calculate the end." Braude was the uncle of E.M.R.'s mother-in-law, a connection we discovered long after the first edition of this book was published!

[6] The best reference is still Ginzel's monumental three-volume work [16], in German. An exceptional survey can be found in the *Encyclopædia of Religion and Ethics* [22, vol. III, pp. 61−141 and vol. V, pp. 835−894]. Useful modern summaries are [6], [12], [40], and [45]; [6] and [40] have extensive bibliographies. The incomparable tables of Schram [41] are the best available for converting dates by hand, whereas those in Parise [36] are best avoided because of an embarrassingly large numbers of errors.