Optimization Methods in Finance

Optimization methods play a central role in financial modeling. This textbook is devoted to explaining how state-of-the-art optimization theory, algorithms, and software can be used to efficiently solve problems in computational finance. It discusses some classical mean–variance portfolio optimization models as well as more modern developments such as models for optimal trade execution and dynamic portfolio allocation with transaction costs and taxes. Chapters discussing the theory and efficient solution methods for the main classes of optimization problems alternate with chapters discussing their use in the modeling and solution of central problems in mathematical finance.

This book will be interesting and useful for students, academics, and practitioners with a background in mathematics, operations research, or financial engineering.

The second edition includes new examples and exercises as well as a more detailed discussion of mean–variance optimization, multi-period models, and additional material to highlight the relevance to finance.

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Optimization Methods in Finance

Second Edition

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Contents

Preface  xi

Part I  Introduction  1

1  Overview of Optimization Models  3
   1.1  Types of Optimization Models  4
   1.2  Solution to Optimization Problems  7
   1.3  Financial Optimization Models  8
   1.4  Notes  10

2  Linear Programming: Theory and Algorithms  11
   2.1  Linear Programming  11
   2.2  Graphical Interpretation of a Two-Variable Example  15
   2.3  Numerical Linear Programming Solvers  16
   2.4  Sensitivity Analysis  17
   2.5  *Duality  20
   2.6  *Optimality Conditions  23
   2.7  *Algorithms for Linear Programming  24
   2.8  Notes  30
   2.9  Exercises  31

3  Linear Programming Models: Asset–Liability Management  35
   3.1  Dedication  35
   3.2  Sensitivity Analysis  38
   3.3  Immunization  38
   3.4  Some Practical Details about Bonds  41
   3.5  Other Cash Flow Problems  44
   3.6  Exercises  47
   3.7  Case Study  51

4  Linear Programming Models: Arbitrage and Asset Pricing  53
   4.1  Arbitrage Detection in the Foreign Exchange Market  53
   4.2  The Fundamental Theorem of Asset Pricing  55
   4.3  One-Period Binomial Pricing Model  56
Contents

4.4 Static Arbitrage Bounds 59
4.5 Tax Clientele Effects in Bond Portfolio Management 63
4.6 Notes 65
4.7 Exercises 65

Part II Single-Period Models 69

5 Quadratic Programming: Theory and Algorithms 71
  5.1 Quadratic Programming 71
  5.2 Numerical Quadratic Programming Solvers 74
  5.3 Sensitivity Analysis 75
  5.4 *Duality and Optimality Conditions 76
  5.5 *Algorithms 81
  5.6 Applications to Machine Learning 84
  5.7 Exercises 87

6 Quadratic Programming Models: Mean–Variance Optimization 90
  6.1 Portfolio Return 90
  6.2 Markowitz Mean–Variance (Basic Model) 91
  6.3 Analytical Solutions to Basic Mean–Variance Models 95
  6.4 More General Mean–Variance Models 99
  6.5 Portfolio Management Relative to a Benchmark 103
  6.6 Estimation of Inputs to Mean–Variance Models 106
  6.7 Performance Analysis 112
  6.8 Notes 115
  6.9 Exercises 115
  6.10 Case Studies 121

7 Sensitivity of Mean–Variance Models to Input Estimation 124
  7.1 Black–Litterman Model 126
  7.2 Shrinkage Estimation 129
  7.3 Resampled Efficiency 131
  7.4 Robust Optimization 132
  7.5 Other Diversification Approaches 133
  7.6 Exercises 135

8 Mixed Integer Programming: Theory and Algorithms 140
  8.1 Mixed Integer Programming 140
  8.2 Numerical Mixed Integer Programming Solvers 143
  8.3 Relaxations and Duality 145
  8.4 Algorithms for Solving Mixed Integer Programs 150
  8.5 Exercises 157
## Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>Mixed Integer Programming Models: Portfolios with Combinatorial Constraints</td>
<td>161</td>
</tr>
<tr>
<td>9.1</td>
<td>Combinatorial Auctions</td>
<td>161</td>
</tr>
<tr>
<td>9.2</td>
<td>The Lockbox Problem</td>
<td>163</td>
</tr>
<tr>
<td>9.3</td>
<td>Constructing an Index Fund</td>
<td>165</td>
</tr>
<tr>
<td>9.4</td>
<td>Cardinality Constraints</td>
<td>167</td>
</tr>
<tr>
<td>9.5</td>
<td>Minimum Position Constraints</td>
<td>168</td>
</tr>
<tr>
<td>9.6</td>
<td>Risk-Parity Portfolios and Clustering</td>
<td>169</td>
</tr>
<tr>
<td>9.7</td>
<td>Exercises</td>
<td>169</td>
</tr>
<tr>
<td>9.8</td>
<td>Case Study</td>
<td>171</td>
</tr>
<tr>
<td>10</td>
<td>Stochastic Programming: Theory and Algorithms</td>
<td>173</td>
</tr>
<tr>
<td>10.1</td>
<td>Examples of Stochastic Optimization Models</td>
<td>173</td>
</tr>
<tr>
<td>10.2</td>
<td>Two-Stage Stochastic Optimization</td>
<td>174</td>
</tr>
<tr>
<td>10.3</td>
<td>Linear Two-Stage Stochastic Programming</td>
<td>175</td>
</tr>
<tr>
<td>10.4</td>
<td>Scenario Optimization</td>
<td>176</td>
</tr>
<tr>
<td>10.5</td>
<td>*The L-Shaped Method</td>
<td>177</td>
</tr>
<tr>
<td>10.6</td>
<td>Exercises</td>
<td>179</td>
</tr>
<tr>
<td>11</td>
<td>Stochastic Programming Models: Risk Measures</td>
<td>181</td>
</tr>
<tr>
<td>11.1</td>
<td>Risk Measures</td>
<td>181</td>
</tr>
<tr>
<td>11.2</td>
<td>A Key Property of CVaR</td>
<td>185</td>
</tr>
<tr>
<td>11.3</td>
<td>Portfolio Optimization with CVaR</td>
<td>186</td>
</tr>
<tr>
<td>11.4</td>
<td>Notes</td>
<td>190</td>
</tr>
<tr>
<td>11.5</td>
<td>Exercises</td>
<td>190</td>
</tr>
<tr>
<td>12</td>
<td>Multi-Period Models: Simple Examples</td>
<td>197</td>
</tr>
<tr>
<td>12.1</td>
<td>The Kelly Criterion</td>
<td>197</td>
</tr>
<tr>
<td>12.2</td>
<td>Dynamic Portfolio Optimization</td>
<td>198</td>
</tr>
<tr>
<td>12.3</td>
<td>Execution Costs</td>
<td>201</td>
</tr>
<tr>
<td>12.4</td>
<td>Exercises</td>
<td>209</td>
</tr>
<tr>
<td>13</td>
<td>Dynamic Programming: Theory and Algorithms</td>
<td>212</td>
</tr>
<tr>
<td>13.1</td>
<td>Some Examples</td>
<td>212</td>
</tr>
<tr>
<td>13.2</td>
<td>Model of a Sequential System (Deterministic Case)</td>
<td>214</td>
</tr>
<tr>
<td>13.3</td>
<td>Bellman’s Principle of Optimality</td>
<td>215</td>
</tr>
<tr>
<td>13.4</td>
<td>Linear–Quadratic Regulator</td>
<td>216</td>
</tr>
<tr>
<td>13.5</td>
<td>Sequential Decision Problem with Infinite Horizon</td>
<td>218</td>
</tr>
<tr>
<td>13.6</td>
<td>Linear–Quadratic Regulator with Infinite Horizon</td>
<td>219</td>
</tr>
<tr>
<td>13.7</td>
<td>Model of Sequential System (Stochastic Case)</td>
<td>221</td>
</tr>
<tr>
<td>13.8</td>
<td>Notes</td>
<td>222</td>
</tr>
<tr>
<td>13.9</td>
<td>Exercises</td>
<td>222</td>
</tr>
</tbody>
</table>
## Contents

### 14 Dynamic Programming Models: Multi-Period Portfolio Optimization
- 14.1 Utility of Terminal Wealth 225
- 14.2 Optimal Consumption and Investment 227
- 14.3 Dynamic Trading with Predictable Returns and Transaction Costs 228
- 14.4 Dynamic Portfolio Optimization with Taxes 230
- 14.5 Exercises 234

### 15 Dynamic Programming Models: the Binomial Pricing Model
- 15.1 Binomial Lattice Model 238
- 15.2 Option Pricing 238
- 15.3 Option Pricing in Continuous Time 244
- 15.4 Specifying the Model Parameters 245
- 15.5 Exercises 246

### 16 Multi-Stage Stochastic Programming
- 16.1 Multi-Stage Stochastic Programming 248
- 16.2 Scenario Optimization 250
- 16.3 Scenario Generation 255
- 16.4 Exercises 259

### 17 Stochastic Programming Models: Asset–Liability Management
- 17.1 Asset–Liability Management 262
- 17.2 The Case of an Insurance Company 263
- 17.3 Option Pricing via Stochastic Programming 265
- 17.4 Synthetic Options 270
- 17.5 Exercises 273

### Part IV Other Optimization Techniques

#### 18 Conic Programming: Theory and Algorithms
- 18.1 Conic Programming 277
- 18.2 Numerical Conic Programming Solvers 282
- 18.3 Duality and Optimality Conditions 282
- 18.4 Algorithms 284
- 18.5 Notes 287
- 18.6 Exercises 287

#### 19 Robust Optimization
- 19.1 Uncertainty Sets 289
- 19.2 Different Flavors of Robustness 290
- 19.3 Techniques for Solving Robust Optimization Models 294
- 19.4 Some Robust Optimization Models in Finance 297
- 19.5 Notes 302
- 19.6 Exercises 302
Contents

20 Nonlinear Programming: Theory and Algorithms 305
  20.1 Nonlinear Programming 305
  20.2 Numerical Nonlinear Programming Solvers 306
  20.3 Optimality Conditions 306
  20.4 Algorithms 308
  20.5 Estimating a Volatility Surface 315
  20.6 Exercises 319

Appendices 321

Appendix Basic Mathematical Facts 323
  A.1 Matrices and Vectors 323
  A.2 Convex Sets and Convex Functions 324
  A.3 Calculus of Variations: the Euler Equation 325

References 327

Index 334
The use of sophisticated mathematical tools in modern finance is now commonplace. Researchers and practitioners routinely run simulations or solve differential equations to price securities, estimate risks, or determine hedging strategies. Some of the most important tools employed in these computations are optimization algorithms. Many computational finance problems ranging from asset allocation to risk management, from option pricing to model calibration, can be solved by optimization techniques. This book is devoted to explaining how to solve such problems efficiently and accurately using the state of the art in optimization models, methods, and software.

Optimization is a mature branch of applied mathematics. Typical optimization problems have the goal of allocating limited resources to alternative activities in order to maximize the total benefit obtained from these activities. Through decades of intensive and innovative research, fast and reliable algorithms and software have become available for many classes of optimization problems. Consequently, optimization is now being used as an effective management and decision-support tool in many industries, including the financial industry.

This book discusses several classes of optimization problems encountered in financial models, including linear, quadratic, integer, dynamic, stochastic, conic, and nonlinear programming. For each problem class, after introducing the relevant theory (optimality conditions, duality, etc.) and efficient solution methods, we discuss several problems of mathematical finance that can be modeled within this problem class.

The second edition includes a more detailed discussion of mean–variance optimization, multi-period models, and additional material to highlight the relevance to finance.

The book’s structure has also been clarified for the second edition; it is now organized in four main parts, each comprising several chapters. Part I guides the reader through the solution of asset liability cash flow matching using linear programming techniques, which are also used to explain asset pricing and arbitrage. Part II is devoted to single-period models. It provides a thorough treatment of mean–variance portfolio optimization models, including derivations of the one-fund and two-fund theorems and their connection to the capital asset pricing model, a discussion of linear factor models that are used extensively
in risk and portfolio management, and techniques to deal with the sensitivity of mean–variance models to parameter estimation. We discuss integer programming formulations for portfolio construction problems with cardinality constraints, and we explain how this is relevant to constructing an index fund. The final chapters of Part II present a stochastic programming approach to modeling measures of risk other than the variance, including the popular value at risk and conditional value at risk.

Part III of the book discusses multi-period models such as the iconic Kelly criterion and binomial lattice models for asset pricing as well as more elaborate and modern models for optimal trade execution, dynamic portfolio optimization with transaction costs and taxes, and asset–liability management. These applications showcase techniques from dynamic and stochastic programming.

Part IV is devoted to more advanced optimization techniques. We introduce conic programming and discuss applications such as the approximation of covariance matrices and robust portfolio optimization. The final chapter of Part IV covers one of the most general classes of optimization models, namely nonlinear programming, and applies it to volatility estimation.

This book is intended as a textbook for Master’s programs in financial engineering, finance, or computational finance. In addition, the structure of chapters, alternating between optimization methods and financial models that employ these methods, allows the book to be used as a primary or secondary text in upper-level undergraduate or introductory graduate courses in operations research, management science, and applied mathematics. A few sections are marked with a ‘∗’ to indicate that the material they contain is more technical and can be safely skipped without loss of continuity.

Optimization algorithms are sophisticated tools and the relationship between their inputs and outputs is sometimes opaque. To maximize the value from using these tools and to understand how they work, users often need a significant amount of guidance and practical experience with them. This book aims to provide this guidance and serve as a reference tool for the finance practitioners who use or want to use optimization techniques.

This book has benefited from the input provided by instructors and students in courses at various institutions. We thank them for their valuable feedback and for many stimulating discussions. We would also like to thank the colleagues who provided the initial impetus for this book and colleagues who collaborated with us on various research projects that are reflected in the book. We especially thank Kathie Cameron, the late Rick Green, Raphael Hauser, John Hooker, Miroslav Karamanov, Mark Koenig, Masakazu Kojima, Vijay Krishnamurthy, Miguel Lejeune, Yanjun Li, François Margot, Ana Margarida Monteiro, Mustafa Pınar, Sebastian Pokutta, Sanjay Srivastava, Michael Trick, and Luís Vicente.