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Flexible Weinstein manifolds

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To Alan Weinstein with admiration.

This survey on flexible Weinstein manifolds, which is essentially an extract from [Cieliebak and Eliashberg 2012], provides to an interested reader a shortcut to theorems on deformations of flexible Weinstein structures and their applications.

1. Introduction

The notion of a *Weinstein manifold* was introduced in [Eliashberg and Gromov 1991], formalizing the symplectic handlebody construction from Alan Weinstein's paper [1991] and the Stein handlebody construction from [Eliashberg 1990]. Since then, the notion of a Weinstein manifold has become one of the central notions in symplectic and contact topology. The existence question for Weinstein structures on manifolds of dimension > 4 was settled in [Eliashberg 1990]. The past five years have brought two major breakthroughs on the uniqueness question: From [McLean 2009] and other work we know that, on any manifold of dimension > 4 which admits a Weinstein structure, there exist infinitely many Weinstein structures that are pairwise nonhomotopic (but formally homotopic). On the other hand, Murphy's h-principle for loose Legendrian knots [Murphy 2012] has led to the notion of *flexible* Weinstein structures, which are unique up to homotopy in their formal class. In this survey, which is essentially an extract from [Cieliebak and Eliashberg 2012], we discuss this uniqueness result and some of its applications.

1A. *Weinstein manifolds and cobordisms.*

Definition. A *Weinstein structure* on an open manifold V is a triple (ω, X, ϕ), where

- ω is a symplectic form on V,
- $\phi : V \to \mathbb{R}$ is an exhausting generalized Morse function,
- X is a complete vector field which is Liouville for ω and gradient-like for ϕ.

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The quadruple (V, ω, X, ϕ) is then called a *Weinstein manifold*.

Let us explain all the terms in this definition. A *symplectic form* is a nondegenerate closed 2-form ω. A *Liouville field* for ω is a vector field X satisfying $L_X\omega = \omega$; by Cartan's formula, this is equivalent to saying that the associated *Liouville form*

$$\lambda := i_X\omega$$

satisfies $d\lambda = \omega$. A function $\phi : V \to \mathbb{R}$ is called *exhausting* if it is proper (i.e., preimages of compact sets are compact) and bounded from below. It is called *Morse* if all its critical points are nondegenerate, and *generalized Morse* if its critical points are either nondegenerate or *embryonic*, where the latter condition means that in some local coordinates $x_1, \dots, x_m$ near the critical point p the function looks like the function ϕ_0 in the *birth–death family*

$$\phi_t(x) = \phi_t(p) \pm tx_1 + x_1^3 - \sum_{i=2}^{k} x_i^2 + \sum_{j=k+1}^{m} x_j^2.$$

A vector field X is called *complete* if its flow exists for all times. It is called *gradient-like* for a function ϕ if

$$d\phi(X) \geq \delta(|X|^2 + |d\phi|^2),$$

where $\delta : V \to \mathbb{R}_+$ is a positive function and the norms are taken with respect to any Riemannian metric on V. Note that away from critical points this just means $d\phi(X) > 0$. Critical points p of ϕ agree with zeroes of X, and p is nondegenerate (resp. embryonic) as a critical point of ϕ if and only if it is nondegenerate (resp. embryonic) as a zero of X. Here a zero p of a vector field X is called embryonic if X agrees near p, up to higher order terms, with the gradient of a function having p as an embryonic critical point.

It is not hard to see that any Weinstein structure (ω, X, ϕ) can be perturbed to make the function ϕ Morse. However, in 1-parameter families of Weinstein structures embryonic zeroes are generically unavoidable. Since we wish to study such families, we allow for embryonic zeroes in the definition of a Weinstein structure.

We will also consider Weinstein structures on a *cobordism*, that is, a compact manifold W with boundary $\partial W = \partial_+ W \amalg \partial_- W$. The definition of a *Weinstein cobordism* (W, ω, X, ϕ) differs from that of a Weinstein manifold only in replacing the condition that ϕ is exhausting by the requirement that $\partial_\pm W$ are regular level sets of ϕ with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$, and completeness of X by the condition that X points inward along $\partial_- W$ and outward along $\partial_+ W$.

A Weinstein cobordism with $\partial_- W = \varnothing$ is called a *Weinstein domain*. Thus any Weinstein manifold (V, ω, X, ϕ) can be exhausted by Weinstein domains $W_k = \{\phi \le c_k\}$, where $c_k \nearrow \infty$ is a sequence of regular values of the function ϕ.

The Liouville form $\lambda = i_X \omega$ induces contact forms $\alpha_c := \lambda|_{\Sigma_c}$ and contact structures $\xi_c := \ker(\alpha_c)$ on all regular level sets $\Sigma_c := \phi^{-1}(c)$ of ϕ. In particular, the boundary components of a Weinstein cobordism carry contact forms which make $\partial_+ W$ a symplectically convex and $\partial_- W$ a symplectically concave boundary (i.e., the orientation induced by the contact form agrees with the boundary orientation on $\partial_+ W$ and is opposite to it on $\partial_- W$). Contact manifolds which appear as boundaries of Weinstein domains are called *Weinstein fillable*.

A Weinstein manifold (V, ω, X, ϕ) is said to be of *finite type* if ϕ has only finitely many critical points. By attaching a cylindrical end

$$\left(\mathbb{R}_+ \times \partial W, d(e^r \lambda|_{\partial W}), \frac{\partial}{\partial r}, f(r)\right)$$

(i.e., the positive half of the symplectization of the contact structure on the boundary) to the boundary, any Weinstein domain (W, ω, X, ϕ) can be completed to a finite type Weinstein manifold, called its *completion*. Conversely, any finite type Weinstein manifold can be obtained by attaching a cylindrical end to a Weinstein domain.

Here are some basic examples of Weinstein manifolds:

(1) $\mathbb{C}^n$ with complex coordinates $x_j + iy_j$ carries the canonical Weinstein structure

$$\left(\sum_j dx_j \wedge dy_j, \frac{1}{2}\sum_j \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right), \sum_j (x_j^2 + y_j^2)\right).$$

(2) The cotangent bundle T^*Q of a closed manifold Q carries a canonical Weinstein structure which in canonical local coordinates (q_j, p_j) is given by

$$\left(\sum_j dp_j \wedge dq_j, \sum_j p_j \frac{\partial}{\partial p_j}, \sum_j p_j^2\right).$$

(As it stands, this is not yet a Weinstein structure because $\sum_j p_j^2$ is not a generalized Morse function, but a perturbation can easily be constructed to make the function Morse.)

(3) The product of two Weinstein manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ has a canonical Weinstein structure $(V_1 \times V_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$. The product $V \times \mathbb{C}$ with its canonical Weinstein structure is called the *stabilization* of the Weinstein manifold (V, ω, X, ϕ).

In a Weinstein manifold (V, ω, X, ϕ), there is an intriguing interplay between Morse theoretic properties of ϕ and symplectic geometry: the stable manifold W_p^- (with respect to the vector field X) of a critical point p is *isotropic* in the symplectic sense (i.e., $\omega|_{W_p^-} = 0$), and its intersection with every regular level set $\phi^{-1}(c)$ is *isotropic* in the contact sense (i.e., it is tangent to ξ_c). In particular, the Morse indices of critical points of ϕ are $\leq \frac{1}{2} \dim V$.

1B. *Stein–Weinstein–Morse.* Weinstein structures are related to several other interesting structures as shown in the following diagram:

$$\begin{array}{ccccc} \mathfrak{Stein} & \xrightarrow{\mathfrak{W}} & \mathfrak{Weinstein} & \xrightarrow{\mathfrak{M}} & \mathfrak{Morse} \\ & & \downarrow & & \\ & & \mathfrak{Liouville}. & & \end{array}$$

Here $\mathfrak{Weinstein}$ denotes the space of Weinstein structures and $\mathfrak{Morse}$ the space of generalized Morse functions on a fixed manifold V or a cobordism W. As before, we require the function ϕ to be exhausting in the manifold case, and to have $\partial_\pm W$ as regular level sets with $\phi|_{\partial_- W} = \min \phi$ and $\phi|_{\partial_+ W} = \max \phi$ in the cobordism case. The map $\mathfrak{M} : \mathfrak{Weinstein} \to \mathfrak{Morse}$ is the obvious one $(\omega, X, \phi) \mapsto \phi$.

The space $\mathfrak{Liouville}$ of *Liouville structures* consists of pairs (ω, X) of a symplectic form ω and a vector field X (the *Liouville field*) satisfying $L_X \omega = \omega$. Moreover, in the cobordism case we require that the Liouville field X points inward along $\partial_- W$ and outward along $\partial_+ W$, and in the manifold case we require that X is complete and there exists an exhaustion $V_1 \subset V_2 \subset \cdots$ of $V = \cup_k V_k$ by compact sets with smooth boundary ∂V_k along which X points outward. The map $\mathfrak{Weinstein} \to \mathfrak{Liouville}$ sends (ω, X, ϕ) to (ω, X). Note that to each Liouville structure (ω, X) we can associate the *Liouville form* $\lambda := i_X \omega$, and (ω, X) can be recovered from λ by the formulas $\omega = d\lambda$ and $i_X d\lambda = \lambda$.

The space $\mathfrak{Stein}$ of *Stein structures* consists of pairs (J, ϕ) of an integrable complex structure J and a generalized Morse function ϕ (exhausting resp. constant on the boundary components) such that $-dd^{\mathbb{C}}\phi(v, Jv) > 0$ for all nonzero $v \in TV$, where $d^{\mathbb{C}}\phi := d\phi \circ J$. If (J, ϕ) is a Stein structure, then $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a symplectic form compatible with J. Moreover, the Liouville field X_ϕ defined by

$$i_{X_\phi} \omega_\phi = -d^{\mathbb{C}}\phi$$

is the gradient of ϕ with respect to the Riemannian metric $g_\phi := \omega_\phi(\cdot, J\cdot)$. In the manifold case, completeness of X_ϕ can be arranged by replacing ϕ by $f \circ \phi$ for a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ with $f'' \geq 0$ and $\lim_{x \to \infty} f'(x) = \infty$; we will suppress the function f from the notation. So we have a canonical map

$$\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}, \quad (J, \phi) \mapsto (\omega_\phi, X_\phi, \phi).$$

It is interesting to compare the homotopy types of these spaces. For simplicity, let us consider the case of a compact domain W and equip all spaces with the C^∞ topology. The results which we discuss below remain true in the manifold case, but one needs to define the topology more carefully; see Section 4C. Since all the spaces have the homotopy types of CW complexes, any weak homotopy equivalence between them is a homotopy equivalence.

The spaces $\mathfrak{Liouville}$ and $\mathfrak{Weinstein}$ are very different: there exist many examples of Liouville domains that admit no Weinstein structure, and of contact manifolds that bound a Liouville domain but no Weinstein domain. The first such example was constructed in [McDuff 1991]: the manifold $[0, 1] \times \Sigma$, where Σ is the unit cotangent bundle of a closed oriented surface of genus > 1, carries a Liouville structure, but its boundary is disconnected and hence cannot bound a Weinstein domain. Many more such examples are discussed in [Geiges 1994].

By contrast, the spaces of Stein and Weinstein structures turn out to be closely related. One of the main results of [Cieliebak and Eliashberg 2012] is this:

Theorem 1.1. *The map $\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}$ induces an isomorphism on π_0 and a surjection on π_1.*

It lends evidence to the conjecture that $\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}$ is a homotopy equivalence.

The relation between the spaces $\mathfrak{Morse}$ and $\mathfrak{Weinstein}$ is the subject of this article. Note first that, since for a Weinstein domain (W, ω, X, ϕ) of real dimension $2n$ all critical points of ϕ have index $\le n$, one should only consider the subset $\mathfrak{Morse}_n \subset \mathfrak{Morse}$ of functions all of whose critical points have index $\le n$. Moreover, one should restrict to the subset $\mathfrak{Weinstein}_\eta^{\text{flex}} \subset \mathfrak{Weinstein}$ of Weinstein structures (ω, X, ϕ) with ω in a fixed given homotopy class η of nondegenerate 2-forms which are *flexible* in the sense of Section 2 below. The following sections are devoted to the proof of the next theorem.

Theorem 1.2 [Cieliebak and Eliashberg 2012]. *Let η be a nonempty homotopy class of nondegenerate 2-forms on a domain or manifold of dimension $2n > 4$. Then*:

(a) *Any Morse function $\phi \in \mathfrak{Morse}_n$ can be lifted to a flexible Weinstein structure (ω, X, ϕ) with $\omega \in \eta$.*

(b) *Given two flexible Weinstein structures (ω_0, X_0, ϕ_0) and (ω_1, X_1, ϕ_1) in $\mathfrak{Weinstein}_\eta^{\text{flex}}$, any path $\phi_t \in \mathfrak{Morse}_n$, $t \in [0, 1]$, connecting ϕ_0 and ϕ_1 can be lifted to a path of flexible Weinstein structures (ω_t, X_t, ϕ_t) connecting (ω_0, X_0, ϕ_0) and (ω_1, X_1, ϕ_1).*

In other words, the map $\mathfrak{M} : \mathfrak{Weinstein}_\eta^{\text{flex}} \to \mathfrak{Morse}_n$ has the following properties:

- $\mathfrak{M}$ is surjective;
- the fibers of $\mathfrak{M}$ are path connected;
- $\mathfrak{M}$ has the path lifting property.

This motivates the following:

Conjecture. On a domain or manifold of dimension $2n > 4$, the map

$$\mathfrak{M} : \mathfrak{Weinstein}_\eta^{\text{flex}} \to \mathfrak{Morse}_n$$

is a Serre fibration with contractible fibers.

2. Flexible Weinstein structures

Roughly speaking, a Weinstein structure is "flexible" if all its attaching spheres obey an h-principle. More precisely, note that each Weinstein manifold or cobordism can be cut along regular level sets of the function into Weinstein cobordisms that are elementary in the sense that there are no trajectories of the vector field connecting different critical points. An elementary $2n$-dimensional Weinstein cobordism (W, ω, X, ϕ), $n > 2$, is called *flexible* if the attaching spheres of all index n handles form in $\partial_- W$ a *loose* Legendrian link in the sense of Section 2C below. A Weinstein cobordism or manifold structure (ω, X, ϕ) is called flexible if it can be decomposed into elementary flexible cobordisms.

A $2n$-dimensional Weinstein structure (ω, X, ϕ), $n \geq 2$, is called *subcritical* if all critical points of the function ϕ have index $< n$. In particular, any subcritical Weinstein structure in dimension $2n > 4$ is flexible.

The notion of flexibility can be extended to dimension 4 as follows. We call a 4-dimensional Weinstein cobordism *flexible* if it is either subcritical, or the contact structure on $\partial_- W$ is overtwisted (or both); see Section 2B below. In particular, a 4-dimensional Weinstein *manifold* is then flexible if and only if it is subcritical.

Remark 2.1. The property of a Weinstein structure being subcritical is not preserved under Weinstein homotopies because one can always create index n critical points (see Proposition 4.7 below). We do not know whether flexibility is preserved under Weinstein homotopies. In fact, it is not even clear to us whether every decomposition of a flexible Weinstein cobordism W into elementary cobordisms consists of flexible elementary cobordisms. Indeed, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of W into elementary cobordisms and $\mathcal{P}_2$ is finer than $\mathcal{P}_1$, then flexibility of $\mathcal{P}_1$ implies flexibility of $\mathcal{P}_2$ (in particular the partition for which each elementary cobordism contains only one critical value is then flexible), but we do not know whether flexibility of $\mathcal{P}_2$ implies flexibility of $\mathcal{P}_1$.

The remainder of this section is devoted to the definition of loose Legendrian links and a discussion of the relevant h-principles.

2A. ***Gromov's h-principle for subcritical isotropic embeddings.*** Consider a contact manifold $(M, \xi = \ker\alpha)$ of dimension $2n-1$ and a manifold Λ of dimension $k-1 \le n-1$. A *monomorphism* $F : T\Lambda \to TM$ is a fiberwise injective bundle homomorphism covering a smooth map $f : \Lambda \to M$. It is called *isotropic* if it sends each $T_x\Lambda$ to a symplectically isotropic subspace of $\xi_{f(x)}$ (with respect to the symplectic form $d\alpha|_\xi$). A *formal isotropic embedding* of Λ into (M, ξ) is a pair (f, F^s), where $f : \Lambda \hookrightarrow M$ is a smooth embedding and $F^s : T\Lambda \to TM$, $s \in [0, 1]$, is a homotopy of monomorphisms covering f that starts at $F^0 = df$ and ends at an isotropic monomorphism $F^1 : T\Lambda \to \xi$. In the case $k = n$ we also call this a *formal Legendrian embedding*.

Any genuine isotropic embedding can be viewed as a formal isotropic embedding $(f, F^s \equiv df)$. We will not distinguish between an isotropic embedding and its canonical lift to the space of formal isotropic embeddings. A homotopy of formal isotropic embeddings (f_t, F_t^s), $t \in [0, 1]$, will be called a *formal isotropic isotopy*. Note that the maps f_t underlying a formal isotropic isotopy form a smooth isotopy.

In the *subcritical* case $k < n$, Gromov proved the following h-principle.

Theorem 2.2 (h-principle for subcritical isotropic embeddings [Gromov 1986; Eliashberg and Mishachev 2002]). *Let (M, ξ) be a contact manifold of dimension $2n-1$ and Λ a manifold of dimension $k-1 < n-1$. Then the inclusion of the space of isotropic embeddings $\Lambda \hookrightarrow (M, \xi)$ into the space of formal isotropic embeddings is a weak homotopy equivalence. In particular*:

(a) *Given any formal isotropic embedding (f, F^s) of Λ into (M, ξ), there exists an isotropic embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is C^0-close to f and formally isotropically isotopic to (f, F^s).*

(b) *Let (f_t, F_t^s), $t \in [0, 1]$, be a formal isotropic isotopy connecting two isotropic embeddings $f_0, f_1 : \Lambda \hookrightarrow M$. Then there exists an isotropic isotopy $\tilde{f}_t$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is C^0-close to f_t and is homotopic to the formal isotopy (f_t, F_t^s) through formal isotropic isotopies with fixed endpoints.*

Let us discuss what happens with this theorem in the critical case $k = n$. Part (a) remains true in all higher dimensions $k = n > 2$:

Theorem 2.3 (existence theorem for Legendrian embeddings for $n > 2$ [Eliashberg 1990; Cieliebak and Eliashberg 2012][1]). *Let (M, ξ) be a contact manifold of dimension $2n-1 \geq 5$ and Λ a manifold of dimension $n-1$. Then given any formal Legendrian embedding (f, F^s) of Λ into (M, ξ), there exists a Legendrian embedding $\tilde{f} : \Lambda \hookrightarrow M$ which is C^0-close to f and formally Legendrian isotopic to (f, F^s).*

Part (b) of Theorem 2.2 does not carry over to the critical case $k = n$: For any $n \geq 2$, there are many examples of pairs of Legendrian knots in $(\mathbb{R}^{2n-1}, \xi_{st})$ which are formally Legendrian isotopic but not Legendrian isotopic; see, for example, [Chekanov 2002; Ekholm et al. 2005].

2B. ***Legendrian knots in overtwisted contact manifolds.*** Finally, we consider Theorem 2.2 in the case $k = n = 2$, that is, for Legendrian knots (or links) in contact 3-manifolds. Recall that in dimension 3 there is a dichotomy between tight and overtwisted contact structures, which was introduced in [Eliashberg 1989]. A contact structure ξ on a 3-dimensional manifold M is called *overtwisted* if there exists an embedded disc $D \subset M$ which is tangent to ξ along its boundary ∂D. Equivalently, one can require the existence of an embedded disc with Legendrian boundary ∂D which is transverse to ξ along ∂D. A disc with such properties is called an *overtwisted disc*.

Part (a) of Theorem 2.2 becomes false for $k = n = 2$ due to Bennequin's inequality. Let us explain this for $\mathbb{R}^3$ with its standard (tight) contact structure $\xi_{st} = \ker \alpha_{st}$, $\alpha_{st} = dz - p\,dq$. To any formal Legendrian embedding (f, F^s) of S^1 into $(\mathbb{R}^3, \xi_{st})$ we can associate two integers as follows. Identifying ξ_{st} to $\mathbb{R}^2$ via the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the (q, p)-plane, the fiberwise injective bundle homomorphism $F^1 : TS^1 \cong S^1 \times \mathbb{R} \to \xi_{st} \cong \mathbb{R}^2$ gives rise to a map $S^1 \to \mathbb{R}^2 \setminus 0$, $t \mapsto F^1(t, 1)$. The winding number of this map around $0 \in \mathbb{R}^2$ is called the *rotation number* $r(f, F^1)$. On the other hand, (F^1, iF^1, ∂_z) defines a trivialization of the bundle $f^*T\mathbb{R}^3$, where i is the standard complex structure on $\xi_{st} \cong \mathbb{R}^2 \cong \mathbb{C}$. Using the homotopy F^s, we homotope this to a trivialization (e_1, e_2, e_3) of $f^*T\mathbb{R}^3$ with $e_1 = \dot{f}$ (unique up to homotopy). The *Thurston–Bennequin invariant* $\mathrm{tb}(f, F^s)$ is the linking number of f with a push-off in direction e_2. It is not hard to see that the pair of invariants (r, tb) yields a bijection between homotopy classes of formal Legendrian embeddings covering a fixed smooth embedding f and $\mathbb{Z}^2$. In particular, the pair (r, tb) can take arbitrary values on formal Legendrian embeddings, while for genuine Legendrian embeddings $f : S^1 \hookrightarrow (\mathbb{R}^3, \xi_{st})$ the

[1] The hypothesis in [Cieliebak and Eliashberg 2012] that Λ is simply connected can be easily removed.

values of (r, tb) are constrained by *Bennequin's inequality* [1983]

$$\mathrm{tb}(f) + |r(f)| \leq -\chi(\Sigma),$$

where Σ is a Seifert surface for f.

Bennequin's inequality, and thus the failure of part (a), carry over to all tight contact 3-manifolds. On the other hand, Bennequin's inequality fails, and except for the C^0-closeness Theorem 2.2 remains true, on overtwisted contact 3-manifolds:

Theorem 2.4 [Dymara 2001; Eliashberg and Fraser 2009]. *Let (M, ξ) be a closed connected overtwisted contact 3-manifold, and $D \subset M$ an overtwisted disc.*

(a) *Any formal Legendrian knot (f, F^s) in M is formally Legendrian isotopic to a Legendrian knot $\tilde{f} : S^1 \hookrightarrow M \setminus D$.*

(b) *Let (f_t, F_t^s), $s, t \in [0, 1]$, be a formal Legendrian isotopy in M connecting two Legendrian knots $f_0, f_1 : S^1 \hookrightarrow M \setminus D$. Then there exists a Legendrian isotopy $\tilde{f}_t : S^1 \hookrightarrow M \setminus D$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is homotopic to (f_t, F_t^s) through formal Legendrian isotopies with fixed endpoints.*

Although Theorem 2.2 (b) generally fails for knots in tight contact 3-manifolds, there are some remnants for special classes of Legendrian knots:

- any two formally Legendrian isotopic *unknots* in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ are Legendrian isotopic [Eliashberg and Fraser 2009];
- any two formally Legendrian isotopic knots become Legendrian isotopic after sufficiently many stabilizations (whose number depends on the knots) [Fuchs and Tabachnikov 1997].

E. Murphy [2012] discovered that the situation becomes much cleaner for $n > 2$: on any contact manifold of dimension ≥ 5 there exists a class of Legendrian knots, called *loose*, which satisfy both parts of Theorem 2.2. Let us now describe this class.

2C. *Murphy's h-principle for loose Legendrian knots.* In order to define loose Legendrian knots we need to describe a local model. Throughout this section we assume $n > 2$.

Consider a Legendrian arc λ_0 in the standard contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ with front projection as shown in Figure 1, for some $a > 0$. Suppose that the slopes at the self-intersection point, as well as at end points of the interval are ± 1, and the slope is everywhere in the interval $[-1, 1]$, so the Legendrian arc λ_0

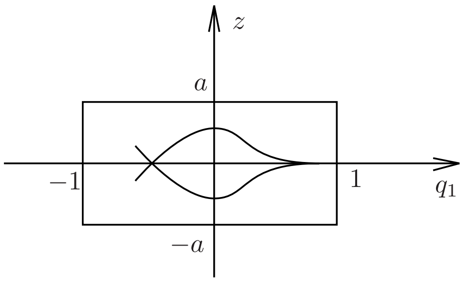

Figure 1. Front of the Legendrian arc λ_0.

is contained in the box

$$Q_a := \{|q_1|, |p_1| \le 1, |z| \le a\}$$

and $\partial\lambda_0 \subset \partial Q_a$. Take the standard contact space $\left(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i\,dq_i\right)$, which we view as the product of the contact space $(\mathbb{R}^3, dz - p_1 dq_1)$ and the Liouville space $\left(\mathbb{R}^{2n-4}, -\sum_{i=2}^{n-1} p_i\,dq_i\right)$. With $q' := (q_2, \dots, q_{n-1})$ and similarly for p', we set

$$|p'| := \max_{2\le i\le n-2} |p_i| \quad \text{and} \quad |q'| := \max_{2\le i\le n-2} |q_i|.$$

For $b, c > 0$ we define

$$P_{bc} := \{|q'| \le b,\ |p'| \le c\} \subset \mathbb{R}^{2n-4},$$
$$R_{abc} := Q_a \times P_{bc} = \{|q_1|, |p_1| \le 1,\ |z| \le a,\ |q'| \le b,\ |p'| \le c\}.$$

Let the Legendrian solid cylinder $\Lambda_0 \subset \left(\mathbb{R}^{2n-1}, dz - \sum_{i=1}^{n-1} p_i\,dq_i\right)$ be the product of $\lambda_0 \subset \mathbb{R}^3$ with the Lagrangian disc $\{p' = 0, |q'| \le b\} \subset \mathbb{R}^{2n-4}$. Note that $\Lambda_0 \subset R_{abc}$ and $\partial\Lambda_0 \subset \partial R_{abc}$. The front of Λ_0 is obtained by translating the front of λ_0 in the q'-directions; see Figure 2. The pair (R_{abc}, Λ_0) is called a *standard loose Legendrian chart* if

$$a < bc.$$

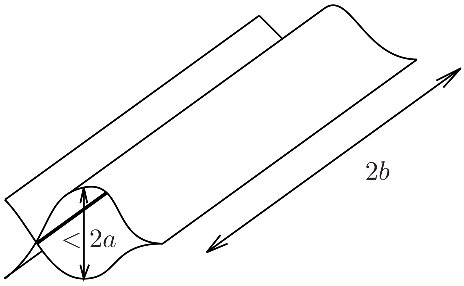

Figure 2. Front of the Legendrian solid cylinder Λ_0.