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978-1-107-05584-1 - Encyclopedia of Mathematics and its Applications: Stochastic Equations
in Infinite Dimensions: Second Edition

Giuseppe Da Prato and Jerzy Zabczyk

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