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Stochastic Equations in Infinite Dimensions

Second Edition

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Contents

Preface

This book is devoted to stochastic evolution equations on infinite dimensional spaces, mainly Hilbert and Banach spaces. These equations are generalizations of Itô stochastic equations introduced in the 1940s by Itô [423] and in a different form by Gikhman [347].

First results on infinite dimensional Itô equations started to appear in the mid-1960s and were motivated by the internal development of analysis and the theory of stochastic processes on the one hand, and by a need to describe random phenomena studied in the natural sciences like physics, chemistry, biology, engineering as well as in finance, on the other hand.

Hilbert space valued Wiener processes and, more generally, Hilbert space valued diffusion processes, were introduced by Gross [363] and Daleckii [183] as a tool to investigate the Dirichlet problem and some classes of parabolic equations for functions of infinitely many variables. An infinite dimensional version of an Ornstein–Uhlenbeck process was introduced by Malliavin [518, 519] as a tool for stochastic study of the regularity of fundamental solutions of deterministic parabolic equations.

Stochastic parabolic type equations appeared naturally in the study of conditional distributions of finite dimensional processes in the form of the so called nonlinear filtering equation derived by Fujisaki, Kallianpur and Kunita [330] and Liptser and Shiryayev [501] or as a linear stochastic equation introduced by Zakaï [737]. Another source of inspiration was provided by the study of stochastic flows defined by ordinary stochastic equations. Such flows are in fact processes with values in an infinite dimensional space of continuous or even more regular mappings acting in a Euclidean space. They are solutions of the corresponding backward and forward stochastic Kolmogorov like equations; see Krylov and Rozovskii [469], Carverhill and Elworthy [146], Kunita [476] and Pardoux [577]. Stochastic partial differential equations constitute an important tool in the analysis of partial differential equations on Hilbert spaces. Such studies are the subject of the monograph [220] by the authors. Let us also mention that the idea of treating delay equations as infinite dimensional processes, systematically used for deterministic equations [258, 392], also proved to be useful for stochastic delay equations (Vinter [698], Chojnowska-Michalik [164]).

As far as applications are concerned, stochastic evolution equations have been motivated by such phenomena as wave propagation in random media (Keller [445], Frish [318]) and turbulence (Novikov [560], Chow [166]). Important motivations came also from biological sciences, in particular from population biology (Dawson [225], Fleming [311]). One has to mention also the early control theoretic applications of Wang [707], Kushner [481] and Bensoussan and Viot [60]. Since the early days, the number of specific equations studied in the literature has increased considerably and we could even say that we are witnessing an explosion of interest in the subject. In particular, stochastic versions of various equations such as reaction-diffusion, wave, beam, Burgers, Musiela, Navier–Stokes, Kardar–Parisi–Zhang, Kuramoto–Sivashinsky, Cahn–Hilliard, Korteweg–de Vries, Schrödinger, Landau–Lifshitz–Gilbert, to mention only a few, have been the subject of numerous studies. In this book we treat only some of them. However, descriptions and bibliographical comments on most of them are given in Chapters 13 and 14 and in the Introduction which is devoted to motivating examples.

Basic theoretical questions on *existence and uniqueness* of solutions were asked and answered, under various sets of conditions, in the 1970s and 1980s and are still of great interest today. An important contribution is due to Pardoux, who, in his thesis [576], obtained fundamental results on stochastic nonlinear partial differential equations (PDEs) of monotone type; see also Krylov and Rozovskii [469]. Basic results on weak solutions are due to Viot; see his thesis [695], and papers [696, 697]. Early important contributions are also due to Bensoussan and Temam [58, 59] and Dawson [226]. More recently, interesting results have been obtained on SPDEs with *random boundary conditions*. For first contributions see Sowers [654] and Da Prato and Zabczyk [218]. Early papers used the Wiener process as a model of noise and stochastic perturbations. The number of studies devoted to equations with *different noise processes* is increasing. In particular, equations with *fractional Brownian motion* and with *Lévy processes* are attracting much attention; see Duncan, Maslowski and Duncan-Pasik [269, 270], Maslowski and Nualart [534] and the recent monograph by Peszat and Zabczyk [596]. Important contributions on numerical solutions have been published. New approaches and original points of view like *Hida's white noise* approach, the *rough paths* approach or *Wiener chaos expansions*, are appearing. They are all discussed in Chapter 14.

The aim of this book is to present basic results on stochastic evolution equations in a rather systematic and self-contained way. We discuss topics covered traditionally by books on ordinary stochastic differential equations: stochastic calculus, existence and uniqueness results, continuous and regular dependence on initial data, Markov property, equations for transition probabilities of Kolmogorov type, absolute continuity of laws induced by solutions on the spaces of trajectories, and asymptotic properties.

The book systematically uses the theory of linear semigroups. Semigroup theory is an important part of mathematics, having several connections with the theory of partial differential equations. Semigroups have been successfully applied to treat semilinear equations; see for instance [401, 512]. The assumption, which we will often make in this book, that the linear part of the equation is the infinitesimal generator of a linear semigroup, is equivalent to the minimal requirement that the equation under study, in its simplest form, has a unique solution continuously depending on the initial data. The semigroup formulation allows a uniform treatment of parabolic, hyperbolic and delay equations. In numerous situations results obtained by more specialized PDE methods can be recovered by the semigroup approach. Early contributions using that approach include some sections of the book by Balakrishnan [32], papers by Curtain and Falb [176] and by Métivier and Pistone [543], and the thesis by Chojnowska-Michalik [163].

A different method for studying stochastic partial differential equations, the so called *variational approach*, was introduced by Pardoux [575] and Krylov and Rozovskii [469]. We do not treat this method in this book. For a recent presentation see the monograph by Prévot and Röckner [602].

In several parts of the book an important role is played by control theory. In particular, control theoretic results are used in the study of transition semigroups, invariant measures and large deviations.

The book is divided into three main parts devoted respectively to foundations of the theory, existence and uniqueness results, and properties of solutions. Analytical results needed in the book, not always easily available in the literature, are gathered in the appendices. Appendix A is devoted to the semigroup treatment of linear deterministic evolutionary problems, so can be regarded as a kind of introduction to the book. Appendices B, C and D concern, respectively, control theory, nuclear and Hilbert–Schmidt operators, and dissipative mappings.

In Part I we recall the measure theoretic foundations of probability theory and give a self-contained exposition of the basic properties of probability measures on separable Banach and Hilbert spaces, needed in what follows. In particular, we prove the Fernique theorem on exponential moments of Gaussian measures, the Bochner characterization of measures on Hilbert spaces, and the Feldman–Hajek theorem on absolute continuity of Gaussian measures. We also introduce reproducing kernels of Gaussian measures and apply this concept to expansions of white noise. In Chapter 3 we list commonly used concepts and theorems from the theory of stochastic processes. We take for granted several results on finite dimensional stochastic processes, in particular classical martingale inequalities. We introduce infinite dimensional Wiener processes and analyze a specific case including spatially homogeneous ones. Finally, we construct the stochastic integral with respect to infinite dimensional Wiener processes and establish Itô's formula and the stochastic Fubini theorem. Maximal inequalities for stochastic integrals are treated in a detailed way.

In Part II we proceed to the main subject of the book, stochastic equations of the form

$$dX = (AX + F(X))dt + B(X)dW(t), \quad x(0) = x, \tag{1}$$

where A is a linear operator, generally unbounded, acting on a Hilbert space H, and F and B are nonlinear, in general discontinuous mappings acting on appropriate spaces. Moreover W is a Wiener process on a Hilbert space U and $x \in H$.

Part II is devoted to existence and uniqueness of solutions. In Chapter 5 we set $F = 0$, B constant and establish existence of weak solutions. We elaborate on the factorization method introduced in [205] and use it to prove the time continuity of the weak solution under broad conditions. Continuity with respect to spatial variables is treated as well. Distribution valued Ornstein–Uhlenbeck processes are also investigated. We give more refined regularity results in the case when A generates an analytic semigroup. In Chapter 6, F is again 0 but B is linear. We first derive sharp estimates for stochastic convolution

$$W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW(s), \quad t \geq 0,$$

where Φ is an operator valued process and $S(\cdot)$ is the contraction semigroup generated by A. We deal with estimates in a wide class of Banach spaces and also include some maximal regularity results from [206]. With good estimates in hand, we establish existence of solutions to (1) by a fixed point argument. We also present a method, applicable only in special situations, of transforming the Itô equation into a deterministic one with random coefficients, which can be treated pathwise. Chapter 7 is devoted to nonlinear equations. We first prove existence and uniqueness when F and B are Lipschitz continuous in H, and then turn to more general B to cover equations with a Nemytskiĭ nonlinearity. The case when F and B are locally Lipschitz continuous or dissipative on a suitable Banach space $E \subset H$ is treated as well. Chapter 8 is devoted to martingale solutions solving the martingale problem, also called weak solutions. We give a proof of the Viot theorem, in the so called compact case, based on the above mentioned factorization method.

Part III of the book is devoted to qualitative properties of solutions. In Chapter 9 we establish continuous dependence of solutions on the initial data and the Feller and Markov properties by an adaptation of finite dimensional methods. We indicate a large class of equations for which the transition semigroup is strongly Feller and for which the Kolmogorov equation can be solved for an arbitrary bounded Borel initial function. The important Bismut–Elworthy–Li formula is derived here and it is used to establish differentiability of solutions to Kolmogorov equations for a wide family of equations. Chapter 10 is on absolute continuity of laws corresponding to solutions of two different equations. We first give a detailed treatment of linear equations based on the Feldman–Hajek theorem. Next we prove the Girsanov theorem and give sufficient conditions for absolute continuity for nonlinear equations. We also establish existence of martingale solutions to equations with irregular drifts by Girsanov's method. Two following chapters concern the asymptotic properties of solutions. Existence and uniqueness of invariant measures and mean square stability are treated first. A careful analysis is carried out for linear equations with an additive and/or multiplicative noise. Nonlinear equations are treated under two types of hypothesis: dissipativity and compactness. Chapter 12 examines the asymptotic properties of solutions when $B(X) = \varepsilon B$, where B is a fixed bounded operator from U to H and ε is small. We establish the large deviation principle for laws of solutions and apply the resulting estimates to the so called exit problem. We generalize finite dimensional results of Freidlin–Wentzell and derive specific asymptotic formulae for so called gradient systems.

As already mentioned, the book covers only basic results of the theory and a number of specific equations are not treated. A comprehensive discussion of the literature and new developments is postponed to Chapters 13 and 14.

In particular we have not covered stochastic equations in nuclear spaces, which have appeared in the study of fluctuation limits of infinite particle systems. They are discussed by Itô [425]; see also Kallianpur and Pérez-Abreu [435]. We do not consider time dependent systems although several extensions to this case are possible. Nor do we discuss quasi-linear equations or equations with stochastic boundary conditions and variational inequalities. Each of those subjects would require several additional chapters. For the same reason we do not report on recent results on the corresponding Fokker–Planck equations, the theory of Dirichlet forms and its applications for solving equations with very irregular coefficients (see Ma and Röckner [516] and the references therein) or on potential theoretic concepts like the Martin boundary (see Föllmer [313]). We do not treat stochastic equations with Lévy noise (see [599] and references therein).

The present book is the second edition of *Stochastic Equations in Infinite Dimensions* published in 1992. We now describe the changes incorporated in the new edition.

There are no major changes in Chapter 1 on random variables or in Chapter 2 on probability measures. We have improved a theorem on white noise expansions, stating it for an arbitrary complete basis in the reproducing kernel. Estimates on the moments of Gaussian measures are derived in more detail and the proof of the Feldman–Hajek theorem is presented, we believe, in a clearer way.

In Chapter 3 we have added the Kolmogorov–Loève–Chentsov theorem on existence of a Hölder continuous version of a random field on bounded open subsets of $\mathbb{R}^d$ with a proof based on the Garsia, Rademich and Rumsay lemma.

In Chapter 4 on stochastic processes we have expanded sections on infinite dimensional Wiener processes. We devote more space to Wiener processes with general, non-trace-class, covariances and discuss specific examples of Wiener processess in $L^2(\mathscr{O})$. We also elaborate an important case of spatially homogeneous Wiener processes. Maximal inequalities for stochastic integrals are treated in a systematic and complete way.

In Chapter 5 we describe in more detail the so called factorization method, a tool to establish time regularity of the solution and existence of invariant measures. At the moment of writing the first edition the method, introduced in the paper [205] by Da Prato, Kwapień and Zabczyk, was fairly new. Since then it has found many applications, some explained in the present edition. In this chapter we also give a new proof of convergence of solutions to equations with Yosida approximations of the linear part of the drift. Existence of regular solutions to equations with higher order operators is treated as well. A section on strong solutions is thoroughly reworked.

A novelty in Chapter 6 is a section on maximal regularity for stochastic convolutions in L^p and $W^{k,p}$ spaces based on a paper by Da Prato and Lunardi [206].

In Chapter 7 on existence and uniqueness of solutions, we add general results which allow us to treat stochastic parabolic equations with Nemytskiĭ diffusion operators. The section on dissipative nonlinearities is extended as well.

Chapter 8 on martingale solutions remains basically as it was.

In Chapter 9 on Markov properties and Kolmogorov equations, we have essentially simplified the proof of differentiability of solutions with respect to initial data. We have also included explicit formulae for higher derivatives of Ornstein–Uhlenbeck transition semigroups. We also expand the section on mild Kolmogorov equations.

Chapters 10–12 are essentially unchanged, although we have tried to simplify the presentation and eliminate some misprints. For more recent results on large time behavior of solutions we refer to our book [220]. Additional results on large deviations can be found in the monograph by Feng and Kurtz [289].

Chapters 13 and 14 are new and are devoted respectively to a survey of results on specific equations and to a description of new developments.

There exist several excellent books on ordinary stochastic differential equations, which provide inspiration for infinite dimensional theory. Earlier books include those by Gikhman and Skorokhod [348], Has'minskii [394], Ikeda and Watanabe [418], Elworthy [280] and more recently Protter [609], Øksendal [567] and Applebaum [23].

Several books on infinite dimensional theory were published before 1992, for example Walsh [702], Belopolskaya and Daleckij [52], Rozovskii [633] and Métivier [542]. More recent are books by Chow [169], Grecksch and Tudor [360], Prévot and Röckner [602], Sanz-Solé [637], Dalang, Khoshnevisan, Mueller, Nualart and Xiao [180], Holden, Øksendal, Ubøe and Zhang [407], Peszat and Zabczyk [599], Kotelenez [458], Veraar [693], internet lecture notes by Hairer [385] and van Neerven's Internet seminar [686]. They all emphasize different aspects of the theory.

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