Introduction: motivating examples

As mentioned in the Preface, stochastic evolution equations in infinite dimensions are natural generalizations of stochastic ordinary differential equations and their theory has motivations coming from both mathematics and natural sciences: physics, chemistry, biology and mathematical finance.

We present here several examples of stochastic equations of the form

$$dX = (AX + F(X))dt + B(X)dW(t), \quad x(0) = x, \tag{1}$$

together with some comments concerning their derivation. Examples 0.1–0.3 have purely mathematical motivations, examples 0.4–0.6 come from physics, 0.7 from chemistry, 0.8–0.9 from biology and 0.10 from finance.

0.1 Lifts of diffusion processes

Consider an ordinary stochastic differential equation on $\mathbb{R}^d$ of the form

$$\begin{cases} dy = f(y)dt + \sum_{j=1}^{N} b_j(y)d\beta_j, \\ y(0) = \xi \in \mathbb{R}^d, \end{cases} \tag{2}$$

where f and $b_1, \dots, b_N$ are continuous mappings from $\mathbb{R}^d$ into $\mathbb{R}^d$. Let us fix a closed subset $K \subset \mathbb{R}^d$ and let E be a Hilbert space of mappings from K into $\mathbb{R}^d$ contained in the space $C(K;\mathbb{R}^d)$ of continuous mappings from K into $\mathbb{R}^d$. The following equation on E

$$\begin{cases} dX = F(X)dt + \sum_{j=1}^{N} B_j(X)d\beta_j \\ X(0) = x \in E \end{cases} \tag{3}$$

in which

$$F(x)(\xi) = f(x(\xi)), \quad B_j(x)(\xi) = b_j(x(\xi)), \quad \xi \in K, \tag{4}$$

is called the *lift* of (2) to E.

Note that if the identity mapping $I_d(\xi): I_d(\xi) = \xi \in K$ belongs to E and there exists a solution to (3) with $x = I_d$ then the formula

$$y(t,\xi) = X(t, I_d)(\xi), \quad \xi \in K \tag{5}$$

defines a version of (2) depending continuously on the initial condition ξ. Such versions are called *stochastic flows*. If in addition the space E consists of diffeomorphisms, then the stochastic flow (4) is the flow of diffeomorphisms. This way one can obtain basic results about stochastic flows from elementary facts on stochastic equations with values in infinite dimensional spaces and known results about Sobolev spaces. See [146] for a detailed exposition of the subject.

0.2 Markovian lifting of stochastic delay equations

A different type of lifting proved to be very useful in the study and applications of stochastic delay equations of the form

$$\begin{cases} dy(t) = \left(\int_{-r}^{0} a(d\theta) y(t+\theta) + f(y(t)) \right) dt + \sum_{j=1}^{N} b_j(y(t)) d\beta_j(t), \\ y(0) = x_0 \in \mathbb{R}^n, \quad y(\theta) = x_1(\theta), \quad \theta \in [-r, 0], \end{cases} \tag{6}$$

where $a(\cdot)$ is an $n \times n$ matrix valued finite measure on $[-r, 0]$ and f and b_j are as in the preceeding example. It turns out that if we define $H = \mathbb{R}^n \times L^2(-r, 0, \mathbb{R}^n)$ then the H-valued process $X(\cdot)$

$$X(t) = \begin{pmatrix} y(t) \\ y_t(\cdot) \end{pmatrix}$$

where $y_t(\theta) = y(t+\theta),\ t \geq 0,\ \theta \in [-r, 0]$, is a solution of the equation

$$\begin{cases} dX = (AX + F(X))dt + \sum_{j=1}^{N} B_j(X) d\beta_j, \\ X(0) = \begin{pmatrix} x_0 \\ x_1(\cdot) \end{pmatrix} \in H, \end{cases} \tag{7}$$

with operators A, F, B_j defined as follows.

The operator A is linear and unbounded with the domain

$$D(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in H : \ x = y(0),\ \ y \in W^{1.2}(-r, 0; \mathbb{R}^n) \right\}$$

and is given by the formula

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_{-r}^{0} a(d\theta) y(\theta) \\ \frac{dy}{d\theta} \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in D(A).$$

Moreover

$$F\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}f(x)\\0\end{pmatrix},\quad \begin{pmatrix}x\\y\end{pmatrix}\in H$$

$$B_j\begin{pmatrix}x\\y\end{pmatrix}u=\begin{pmatrix}b_j(x)\\0\end{pmatrix}u,\quad \begin{pmatrix}x\\y\end{pmatrix}\in H,\quad u\in\mathbb{R}^1,\ j=1,\dots,N.$$

Conversely, under fairly general conditions, the $\mathbb{R}^n$-dimensional coordinate of the solution X of equation (7) is a solution of the stochastic equation (6). The main advantage of dealing with equation (7) rather than with (6) is that the solution of (7) is Markovian and the solution of (6) is not. For more details and applications we refer to [164, 258, 698].

0.3 Zakaï's equation

Let y be the solution of the equation in $\mathbb{R}^n$

$$dy(t)=f(y(t))dt+dW(t)+BdV(t),\quad y(0)=\xi\in\mathbb{R}^n$$

and let γ be the "observation"

$$\gamma(t)=\int_0^t g(y(s))ds+V(t),\quad t\geq 0, \tag{8}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$, $g:\mathbb{R}^n\to\mathbb{R}^m$ are given mappings, $B=(b_{j,k})$ is an $n\times m$ matrix and W, V are independent Wiener processes of dimensions n and m respectively.

It is of great interest to describe the evolution in time of the conditional distribution μ_t of $y(t)$ with respect to the σ-field generated by $\gamma(s),\ s\leq t,\ t\geq 0$. One approach to the problem was proposed by Zakaï [737] and is related to the so called *Zakaï* equation:

$$\begin{cases} dX(t,x)=\left[\dfrac{1}{2}\displaystyle\sum_{j=1}^n\frac{\partial^2 X}{\partial x_j^2}(t,x)-\sum_{j=1}^n\frac{\partial X}{\partial x_j}(t,x)f_j(x)\right]dt\\ \qquad\qquad +X(t,x)\displaystyle\sum_{k=1}^m g_k(x)d\gamma_k(t)-\sum_{k=1}^m\sum_{j=1}^n\frac{\partial X}{\partial x_j}(t,x)b_{jk}d\gamma_k(t)\\ X(0,x)\ =x\in E\end{cases} \tag{9}$$

on a space of real valued functions defined on $\mathbb{R}^k$. Equation (9) is of the form (1), with differential operators A and B of, respectively, second and first order, and $F=0$. Moreover $\gamma_1,\dots,\gamma_k$ are coordinates of the process γ, which is a Wiener process, under an equivalent probability measure. If there exists a solution $X(\cdot)$ of (9) then, under rather general conditions, the function valued process

$p(t) := X(t)/\langle X(t), 1\rangle,\ t \geq 0$, where

$$\langle X(t), 1\rangle = \int_{\mathbb{R}^k} X(t, \xi) d\xi, \quad t \geq 0,$$

is identical with the process of the densities of the conditional laws $\mu_t,\ t \geq 0$. Since the law $\mathscr{L}(\gamma(\cdot))$ of the process $\gamma(\cdot)$ on $C([0, T]; \mathbb{R}^k)$, T arbitrary positive constant, is equivalent to the law of $\mathscr{L}(V(\cdot))$ so, to study the problem of existence and uniqueness of the solutions to (9) or path properties of $X(\cdot)$, one can assume that processes $\gamma_1, \ldots, \gamma_k$ are independent real valued Wiener processes. In this sense Zakai's equation is of the type (1). For more details about the equation see [434] and [577].

We pass now to examples arising in the natural sciences.

0.4 Random motion of a string

The following model of motion of an elastic string in a viscous random environment was proposed by Funaki [331]. There is a vast literature on related models so we will be more detailed here.

Let us start by remarking that the motion of a particle in a viscous environnement in $\mathbb{R}^d$ under a forcing field $f(y)$, $y \in \mathbb{R}^d$, can be described by the first order equation

$$y' = f(y), \quad y(0) \in \mathbb{R}^d.$$

Fix a natural number $N > 1$ and a sequence $W_1, \ldots, W_N$ of independent d-dimensional Wiener processes and consider a system of N particles that move under the influence of three kinds of forces: *elastic forces* acting between neighboring particles, proportional to the distance between particles, the *external force* f and the *random forces* of white noise type.

The movement of the kth particle is then described by a properly normalized stochastic ordinary differential equation:

$$\begin{aligned} dy_k = &\left[(\kappa/2)N^2(y_{k+1} + y_{k-1} - 2y_k) + f(y_k)\right] dt \\ &+ \sqrt{N}\, b(y_k) dW_k, \quad k = 1, \ldots, N, \end{aligned} \tag{10}$$

where κ is the modulus of the elastic forces and for each y, $b(y)$ is a matrix describing the intensities of the random forces. Assume that transformations f and b are Lipschitz continuous, then the system (10) determines uniquely processes $y_k(t),\ k = 1, \ldots, N,\ t \geq 0$, provided the initial conditions $y_k(0), k = 1, \ldots, N$ are given as well as the processes $y_0(t),\ t \geq 0$, and $y_{N+1}(t),\ t \geq 0$, which appear in the equations describing motion of the first and the Nth particles respectively. Let $\xi_k = \frac{k-1}{N-1},\ k = 1, \ldots, N$, and let $x(\xi),\ \xi \in [0, 1]$, be a fixed continuous function with values in $\mathbb{R}^d$. We fix the initial conditions $y_1(0), \ldots, y_N(0)$ by requiring that

$$y_k(0) = x(\xi_k), \quad k = 1, \ldots, N.$$

The processes y_0 and y_{N+1} are determined by one of the following three boundary conditions

$$y_0(t) = y_1(t), \quad y_{N+1}(t) = y_N(t), \quad t \geq 0, \tag{11}$$

or

$$y_0(t) = y_1(t) = 0, \quad y_{N+1}(t) = y_N(t), \quad t \geq 0, \tag{12}$$

or

$$y_0(t) = y_1(t) = 0, \quad y_{N+1}(t) = y_N(t) = 0, \quad t \geq 0. \tag{13}$$

When considering cases (12) and (13) we will require also that $x_0(0) = 0$ and $x_0(0) = x_0(1) = 0$. The following process $X_N(t, \cdot)$, $t \geq 0$, with values in the function space $E = L^2(0, 1; \mathbb{R}^d)$ or in $E = C([0, 1]; \mathbb{R}^d)$

$$X_N(t, \xi) = y_k(t) + \frac{\xi - \xi_k}{\xi_{k+1} - \xi_k} y_{k+1}(t), \quad \xi \in [\xi_k, \xi_{k+1}], \; k = 1, 2, \ldots, N-1,$$

can be regarded as a discrete approximation of the moving string. Let $\mathscr{L}(X_N)$ be its distribution on $C([0, T]; E)$ ($T > 0$ a fixed number) of the process X_N, $N = 2, 3, \ldots$ The following result is due to Funaki [331]. In its formulation

$$E^2 = \left\{ z \in W^{2,2}([0, 1]; \mathbb{R}^d) : \frac{d^2 z}{d\xi^2} \in E \right\},$$

E^2 stands for the Sobolev space of functions $x \in W^{2,2}([0, 1]; \mathbb{R}^d)$.

Theorem 0.1 (see [331]) *The sequence $\{\mathscr{L}(X_N)\}$ converges weakly on $C([0, T]; E)$ to $\mathscr{L}(X)$, where the process X is a solution of equation* (1) *and the operator $A = d^2/d\xi^2$ with the domain $D(A)$ is equal respectively to*

$$D(A) = E^2 \cap \left\{ z : \frac{dz}{d\xi}(0) = \frac{dz}{d\xi}(1) = 0 \right\},$$

or

$$D(A) = E^2 \cap \left\{ z : z(0) = \frac{dz}{d\xi}(1) = 0 \right\},$$

or

$$D(A) = E^2 \cap \{ z : z(0) = z(1) = 0 \},$$

according to which of the conditions (11), (12), (13) *are considered in the definition of X_N. The process $W(\cdot)$ is a cylindrical Wiener process on $U := L^2(0, 1; \mathbb{R}^1)$ with* *the identity covariance operator. Moreover*

$$F(z)(\xi) = f(z(\xi)), \quad z \in E,\ \xi \in [0,1],$$
$$(B(z)u)(\xi) = b(z(\xi))u(\xi), \quad z \in E,\ u \in U,\ \xi \in [0,1].$$

0.5 Stochastic equation of the free field

Hida and Streit showed in [403] that the equation

$$dX = -(\lambda - \Delta)^{1/2} X dt + dW,$$

where W is again a cylindrical Wiener process on the Hilbert space $E = U = L^2(D)$ with the covariance operator I, has a stationary solution corresponding to the Gaussian invariant measure with the covariance

$$C = (\lambda - \Delta)^{-1/2}.$$

This stationary solution can be interpreted as the Euclidean free field.

0.6 Equation of stochastic quantization

Let D be an arbitrary open subset of $\mathbb{R}^n$ and $r(u, v),\ u, v \in D$, a *positive definite*, continuous function. This means that r is a continuous function such that

$$\sum_{j,k=1}^{n} r(u_j, u_k)\lambda_j\lambda_k \geq 0, \quad \forall\, u_j, u_k \in D,\ \lambda_j, \lambda_k \in \mathbb{R}^1. \tag{14}$$

It is well known that positive definite, continuous functions are precisely the covariance functions of the mean-square continuous, *Gaussian random fields* $\{\xi_u, u \in D\}$, i.e.,

$$r(u, v) = \mathbb{E}(\xi_u \xi_v), \quad u, v \in D. \tag{15}$$

In (15), $\mathbb{E}$ stands for the expectation with respect to the probability measure $\mathbb{P}$ given on $(\Omega, \mathscr{F})$; we assume that $\mathbb{E}(\xi_u) = 0$, $u \in U$. On the other hand, let $H_m,\ m \in \mathbb{N}$, be Hermite polynomials

$$H_m(z) = m! \sum_{n+2k=m} (-1)^k \frac{z^n}{n!k!2^k}, \quad m \in \mathbb{N},\ z \in \mathbb{R}^1 \tag{16}$$

given by the *generating function* formula

$$e^{\lambda z - \frac{1}{2}\lambda^2} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} H_m(z), \quad \lambda, z \in \mathbb{R}^1.$$

Let E be a space of continuous functions defined on D. Given a positive definite, continuous function $r(\cdot,\cdot)$, the so called mth-*Wick power*: $x^m(z) : z \in D$, of an arbitrary function $x \in E$, is defined by the formula

$$:x^m(z) := \sqrt{r^m(z,z)}H_m\left(\frac{x(z)}{\sqrt{r(z,z)}}\right), \quad z \in D,\ x \in E. \tag{17}$$

It follows from (16) that

$$:x^m(z) := m! \sum_{n+2k=m} (-1)^k \frac{x^n(z) r^k(z,z)}{n!k!2^k}, \quad z \in D,\ x \in E. \tag{18}$$

It is therefore clear that for $x \in E$ the formulae (17) and (18) define continuous functions.

The *stochastic quantization equation* is of the form

$$dX = [AX- :X^m:]dt + dW, \tag{19}$$

where the term $:X^m:$, m an odd number, is a fairly irregular drift, called the "Wick power." The equation is important in statistical physics as the invariant measure ν for the solution is, up to a multiplicative constant, of the Gibbs form

$$\nu(dx) = e^{-\frac{1}{m+1}\int_D :x^{m+1}:d\xi}\mu(dx),$$

where μ is the invariant measure for the Ornstein–Uhlenbeck equation

$$dX = AX + dW.$$

W is a cylindrical Wiener process on $U = L^2(D)$, with the covariance I, A is a self-adjoint negative definite operator, generally unbounded on U, and m is an odd number.

To throw some light on the very definition of the Wick power and its connection with Gaussian random fields, assume that γ is a symmetric Gaussian measure on a Hilbert space H with the covariance operator $C = \frac{1}{2}A^{-1}$ of the integral type with the kernel function $r(\cdot,\cdot)$. The following two observations are essential.

Fact 1 Let x be a real valued, symmetric, Gaussian random variable defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let G be the σ- algebra generated by x. Then the random variables $(m!)^{-1/2}H_m(x)$, $m \in \mathbb{N}$, constitute a complete and orthonormal basis in $L^2(\Omega, \mathscr{F}, \mathbb{P})$. They are obtained from x^m, $m \in \mathbb{N}$, by the Hilbert–Schmidt orthogonalization procedure.

Fact 2 Let us assume that the random field $x(u), u \in D$, has continuous realizations belonging the the space E almost surely. Then the law γ of the random field $x(u), u \in D$, is a symmetric Gaussian measure on E equipped with its Borel σ-algebra B. Moreover the functionals $x(z)$, $z \in D$, defined for x in E, can be regarded as random variables on (E, B, γ). Therefore the family $x(z)$, $z \in D$, constitutes a random field with the law γ and the covariance function $r(\cdot,\cdot)$. Consequently the families

$\{(x(z)^m, z \in D\}$, $m = 0, 1, \ldots$ and $\{: x(z)^m :, z \in D\}$ are random fields as well and the latter can be obtained from the former by the orthogonalization procedure sometimes called *renormalization*.

In applications, H is an appropriate Sobolev space and the covariance operator is $C = (\lambda - \Delta)^{-1},\ \lambda \in \mathbb{R}^1$, where λ is a fixed number (in the definition of the Laplace operator one has to take into account boundary conditions). If the operator C obtained this way is nuclear then it determines a Gaussian, symmetric measure γ on H. If the measure is concentrated on the space E of continuous functions then the Wick powers have, according to the previous considerations, well defined meaning as well as the stochastic equations (19). However, in several situations of physical interest the measure γ is not concentrated on a space of continuous functions but on some spaces of distributions on D. Then additional care must be given to the nonlinear term in (19). For more details we refer to the papers [92, 288, 432].

In particular, let $D = [0, \pi]$, $H = L^2(D)$. The operator $\Delta = d^2/dz^2$, with the domain $D(\Delta) = H_0^1(0, \pi) \cap H^2(0, \pi)$, is negative and self-adjoint and the operators $R_\lambda = (\lambda - \Delta)^{-1}$, $\lambda \geq 0$, are nuclear, integral operators with kernels:

$$r_\lambda(u, v) = r_\lambda(v, u) = \frac{\sinh(\lambda^{1/2} v)\sinh(\lambda^{1/2}(\pi - u))}{\lambda^{1/2}\sinh(\lambda^{1/2}\pi)}, \quad 0 \leq u \leq v \leq \pi, \lambda > 0$$

$$r_0(u, v) = r_0(v, u) = \frac{1}{\pi}(v(\pi - u)), \quad 0 \leq u \leq v \leq \pi.$$

One can easily show that stochastic Gaussian processes with covariance functions $r_\lambda(\cdot, \cdot)$, $\lambda > 0$, have continuous versions with values 0 at the boundary points 0 and π. Therefore their laws are Gaussian measures γ_λ, concentrated on the space $E = C_0([0, \pi])$. The measure γ_0 corresponding to $\lambda = 0$ is the law of a *Brownian bridge process*. In this case equation (19) is of the form

$$\begin{cases} dX(t, \xi) = \dfrac{d^2}{dz^2} X(t, \xi) - [X^3(t, \xi) - \dfrac{3}{\pi}(\xi(\pi - \xi))X(t, \xi)]dt + dW, \\ \xi \in [0, \pi], \quad t \geq 0 \\ X(t, 0) = X(t, \pi) = 0, \quad X(0, \cdot) = x \in C_0([0, \pi]). \end{cases} \tag{20}$$

Equations similar to (20) appear in the following two different contexts.

0.7 Reaction-diffusion equation

The deterministic reaction-diffusion equations are of the form

$$\frac{\partial u}{\partial t}(t, \xi) = \sigma^2 \frac{\partial^2 u}{\partial \xi^2}(t, \xi) + f(u(t, \xi)), \quad t \geq 0,\ \xi \in [0, T], \tag{21}$$

with appropriate boundary conditions. The function $(\partial u/\partial t)(t, \xi)$ is the so called *rate function* consisting of the gain term l and the loss term m. In the case of two initial and one final products: $l(u) = au^2 + g$, $m(u) = u^3 + bu$, $u \in \mathbb{R}^1$ and a, b and g are positive constants. The many-particle nature of a real system results in internal fluctuations. This leads to

$$\frac{\partial u}{\partial t}(t, \xi) = \sigma^2 \frac{\partial^2 u}{\partial \xi^2}(t, \xi) + f(u(t, \xi)) + \dot{W}(t, \xi), \quad t \geq 0, \ \xi \in [0, T], \tag{22}$$

where $\dot{W}$ is a temporal and spatial white noise. Equation (22) can be formulated as an equation of type (1). More information about the model can be found in [26] and [520].

0.8 An example arising in neurophysiology

Neurons or nerve cells can be regarded as long thin cylinders, which act much like electrical cables. Let us identify such a cylinder with the interval $[0, L]$. Let $V(t, \xi)$ be the electrical potential at the point ξ at time t. The potential satisfies a nonlinear parabolic equation coupled with a system of ordinary differential equations, called Hodgkin–Huxley equations. We will not write those equations here but indicate only two references, [401] and [540]. Nagumo's equation can be regarded as preliminary to the Hodgkin–Huxley system. In some ranges of the potential the Hodgkin-Huxley equation can be approximated by the cable equation:

$$\frac{\partial V}{\partial t}(t, \xi) = \frac{\partial^2 V}{\partial \xi^2}(t, \xi) - V(t, \xi) + \dot{W}(t, \xi), \quad t \geq 0, \ \xi \in [0, L], \tag{23}$$

where $\dot{W}(t, \xi)$ is the current arriving at ξ at moment t. It is reasonable to conjecture that impulses arrive according to a Poisson process, in both time and space variables. If the intensity of the impulses is high one can assume that $\dot{W}$ is of white noise type. We refer to [702] for more details. In this way one arrives again at a stochastic equation of type (1). More sophisticated arguments suggest that a nonlinear term $p(V)$, where p is a cubic polynomial, should be added to the right hand side of (23); see the paper by McKean [540].

0.9 Equation of population genetics

Stochastic semilinear equations have also been used in population genetics to model changes in the structure of a population in time and in space. In particular, Dawson [225] proposed the following equation

$$dp(t, \xi) = a\Delta p(t, \xi)dt + b\sqrt{p^+(t, \xi)}dW_t, \quad \xi \in \mathbb{R}^d, \tag{24}$$

for the mass distribution $p(t,\cdot)$ of the population at time $t \geq 0$. A slightly different equation

$$dp(t,\xi) = (\Delta p(t,\xi) + ap(t,\xi) - b)dt + b\left(\frac{1}{2}\, p^+(t,\xi)(1 - p(t,\xi)^+)\right)^{1/2} dW_t, \quad \xi \in \mathbb{R}^d, \tag{25}$$

was proposed by Fleming [311]. In equations (24) and (25), a, b are positive constants and W is a U-valued Wiener process with a nuclear covariance operator Q. Both the equations are of type (1). Here $E = L^2(\mathbb{R}^d)$, $A = a\Delta$ or $A = \Delta + aI - c$, $D(A) = H^2(\mathbb{R}^d)$ and

$$(Bx)u(\xi) = b\left(x^+(\xi)\right)^{1/2} u(\xi)$$

or

$$(Bx)u(\xi) = b\left((1/2)x^+(\xi)(1 - x(\xi))^+\right)^{1/2} u(\xi).$$

0.10 Musiela's equation of the bond market

Let $P(t,T)$, $0 \leq t \leq T$, denote the price, at moment t, of a bond which matures at $T \geq t$. It is often represented in the form

$$P(t,T) = e^{-\int_t^T f(t,s)ds},$$

where $f(t,s)$, $0 \leq t \leq s$, is a random field called the "forward rate." In the approach proposed by Heath, Jarrow and Morton [400], one assumes that for every T, the process $f(t,T)$, $t \leq T$, has a representation of the form:

$$\begin{cases} df(t,x) = \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) = f_0(T), \end{cases} \tag{26}$$

where W is a Wiener process, for simplicity assume that it is one dimensional. Musiela [555] noticed that analysis of the forward rates is simplified if they are considered in the "moving frame" resulting in the new parameterization

$$f(t, t+\xi) = r(t,\xi), \quad t \geq 0,\ \xi \geq 0$$

with ξ being the "time to maturity." Then r satisfies the equation

$$dr(t,\xi) = \left(\frac{\partial r}{\partial \xi}(t,\xi) + \widetilde{\alpha}(t,\xi)\right) dt + \widetilde{\sigma}(t,\xi)dW_t$$

where

$$\widetilde{\alpha}(t,\xi) = \alpha(t, t+\xi), \quad \widetilde{\sigma}(t,\xi) = \sigma(t, t+\xi).$$