Part I

How geometric structures arise in supersymmetric field theories

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Geometrical structures in (Q)FT

Part I of this book is introductory in nature. Its purpose is to motivate our geometric approach to supersymmetric field theory. We show how geometric structures arise in classical and quantum field theories on quite general grounds. In Chapter 1 we consider the basic geometric structures which hold independently of supersymmetry. In Chapter 2 we specialize to the supersymmetric case (rigid and local) where more elegant structures emerge. Not being part of the technical body of the book, these chapters are rather elementary and sketchy. However, we show how dualities, modularity, and other stringy patterns are universal features of field theory.

Throughout this book, by a *field theory* we shall mean a *Lagrangian* field theory, that is, a classical or quantum system whose dynamics is described by a Lagrangian $\mathcal{L}$ with no more than two derivatives of the fields.

1.1 (Gauged) σ–models

Most quantum field theories (QFTs) have scalar fields. Usually we can understand a lot about the dynamics of a field theory just by studying its scalar sector. This is *a fortiori* true if the theory has (enough) supersymmetries, since in this case all other sectors are related to the scalar one by a symmetry. The understanding of the scalars' geometry is relevant even for theories, like quantum chromodynamics (QCD), that do not have fundamental scalar fields in their microscopic formulation. At low energy, QCD is well described by an effective scalar model whose fields represent pions (the lightest particles in the hadronic spectrum). Historically, this effective theory was the original σ–model. It encodes all current algebra of QCD, and its phenomenological predictions are quite a success [303, 304, 305]. Our first goal is to generalize this model. We begin by considering a theory with only scalar fields. In the next section we will add fields in arbitrary (finite) representations of the Lorentz group.

1.1.1 The target space $\mathcal{M}$

We consider a general field theory in D space–time dimensions whose Lagrangian description contains only scalar fields which we denote as ϕ^i, with $i = 1, 2, \dots, n$. Let us write down the most general local, Hermitian, Poincaré–invariant Lagrangian having (at most) two derivatives; for $D \neq 2$ it has the form

$$\mathcal{L} = -\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial^\mu \phi^j + \text{terms with no derivative} \tag{1.1}$$

for some (field–dependent) real symmetric matrix $g_{ij}(\phi)$. For $D = 2$ we may add the P and T odd term $b_{ij}(\phi)\, \epsilon^{\mu\nu}\, \partial_\mu \phi^i\, \partial_\nu \phi^j$ with $b_{ij}(\phi)$ antisymmetric. For the moment we limit ourselves to P–invariant models and set $b_{ij} = 0$.

Unitarity requires the kinetic terms to be positive, so $g_{ij}(\phi)$ is a positive–definite matrix. Physical quantities are independent of the fields we use to parametrize the configuration, that is, observables are invariant under field reparametrizations of the form

$$\phi^i \to \varphi^i = \varphi^i(\phi). \tag{1.2}$$

Written in terms of the new fields φ^i, the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2} \widetilde{g}_{ij}(\varphi)\, \partial_\mu \varphi^i\, \partial^\mu \varphi^j + \cdots \tag{1.3}$$

where

$$\widetilde{g}_{ij}(\varphi) \equiv \frac{\partial \phi^k}{\partial \varphi^i}\, g_{kl}\, \frac{\partial \phi^l}{\partial \varphi^j}. \tag{1.4}$$

The above equations have a simple geometric interpretation: the fields ϕ^i are local coordinates on a (smooth) manifold $\mathcal{M}$ and g_{ij} is a Riemannian metric for $\mathcal{M}$, which correctly transforms under diffeomorphisms as a symmetric tensor, Eq. (1.4). This interpretation allows us to describe the situation in more geometric terms: we have two manifolds, the *target* one $\mathcal{M}$, which can have a non–trivial topology,[1] and the space–time manifold Σ (which, for the moment, we take to be just Minkowski space $\mathbb{R}^{D-1,1}$). A classical field configuration is a (smooth) map

$$\Phi \colon \Sigma \to \mathcal{M} \tag{1.5}$$

which in local coordinates is given by the functions $\phi^i(x^\mu)$. The Lagrangian $\mathcal{L}$ is simply the trace (with respect to the space–time metric $\eta_{\mu\nu}$) of the pull–back (i.e., induced) metric $\Phi^* g$.

[1] Hence the fields ϕ^i are, in general, only locally defined on $\mathcal{M}$.

(Q)FTs defined by maps $\Sigma \to \mathcal{M}$ and the Lagrangian

$$\mathcal{L} = -\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial^\mu \phi^j \tag{1.6}$$

are called σ*–models*. We stress again that in such models *all* physical quantities, being reparametrization-independent, should be differential–geometric invariants of the Riemannian manifold $(\mathcal{M}, g)$. This simple observation, which we call the *Geometric Principle,* is quite powerful.

Example: the renormalization group β–functions

To show the power of the Geometric Principle, we discuss the one–loop β–functions of the σ–model. We take $D = 2$, the space–time dimension in which the model is power–counting renormalizable. We may consider the most general Lagrangian of the form (1.1), since the interactions with no derivative (the potential) is a soft term that does not affect the β–functions of the derivative couplings. We introduce the Planck constant, $\hbar$, as a loop–counting device; recall that in perturbative QFT the k–loop contribution to the β–function scales like $\hbar^{k-1}$. The action is

$$S = -\frac{1}{2\hbar} \int_\Sigma g_{ij}\, \partial_\mu \phi^i \partial^\mu \phi^j\, d^2 z. \tag{1.7}$$

We see that a rescaling $\hbar \to \lambda\, \hbar$ is equivalent to $g_{ij} \to \lambda^{-1} g_{ij}$, so the weak-coupling limit $\hbar \to 0$ is just the large volume limit for $\mathcal{M}$,

$$\mathrm{vol}(\mathcal{M}) \propto \hbar^{-\dim \mathcal{M}/2}. \tag{1.8}$$

A general σ–model has an *infinite* number of coupling constants, $g_{i_1 i_2 \cdots i_l}$,

$$S = -\frac{1}{2\hbar} \int d^2 z \sum_{l=2}^{\infty} g_{i_1 i_2 \cdots i_l}\, \phi^{i_3} \phi^{i_4} \cdots \phi^{i_l}\, \partial_\mu \phi^{i_1}\, \partial^\mu \phi^{i_2},$$

namely, the Taylor coefficients[2] of the metric $g_{ij}(\phi)$. We can conveniently assemble the infinite set of β–functions into a symmetric tensor, $\beta_{ij}(\phi)$, whose Taylor coefficients are the β–functions of the couplings $g_{i_1 i_2 \cdots i_l}$. The renormalization group (RG) flow then takes the form

$$\mu \frac{\partial}{\partial \mu} \frac{g_{ij}(\phi)}{\hbar} = \beta_{ij}(\phi). \tag{1.9}$$

[2] Assuming the metric is of class C^ω.

The Geometric Principle implies that $\beta_{ij}(\phi)$ is a covariant symmetric tensor on $\mathcal{M}$ made out of the metric g_{ij} and its derivatives. Moreover, $\beta_{ij}(\phi)$ should vanish for a flat metric, since in that case the QFT is free. Therefore, $\beta_{ij}(\phi)$ is a symmetric tensor which has an expansion as a sum of products of Riemann tensor covariant derivatives, $\nabla_{i_1}\cdots\nabla_{i_s}R^j{}_{klm}$, with the indices contracted in a suitable way using the effective inverse metric $\hbar\, g^{ij}$. Since $\nabla_{i_1}\cdots\nabla_{i_s}R^j{}_{klm}$ is invariant under $g_{ij}\mapsto\hbar^{-1}g_{ij}$, all $\hbar$ dependence arises from the inverse metric contractions. Counting indices to be contracted, we see that a term in this expansion scales with the volume as

$$\hbar^{r+s-1}, \tag{1.10}$$

where r is the number of Riemann tensors and $2s$ the total number of covariant derivatives. Since the one–loop contribution scales as $\hbar^0$, this leaves only one possibility: one Riemann tensor and no derivative. Thus

$$\beta_{ij}|_{\text{one loop}} = c_1\, R_{ij} + c_2\, g_{ij}\, R, \tag{1.11}$$

for some constants c_1, c_2. We claim that $c_2 = 0$ while c_1 is a universal coefficient that does not depend on $\dim\mathcal{M}$. Indeed, take $\mathcal{M}=\mathbb{R}^n\times\mathcal{N}$. The fields of the flat factor are free, and then $\beta_{ij}|_{\mathbb{R}^m}=0$, whereas Eq. (1.11) gives $\beta_{ij}|_{\mathbb{R}^m}=c_2\,R\,\delta_{ij}$. So $c_2=0$. On the other hand, $\beta_{ij}|_{\mathcal{N}}=c_1\,R_{ij}$ cannot depend on $m\equiv$ the number of flat directions, since they correspond to decoupled free fields. Hence c_1 is independent of $\dim\mathcal{M}$, and it may be computed using any convenient manifold.

The fixed points of the RG flow need not correspond to zeros of the β–function, i.e., to metrics such that $\beta_{ij}=0$; more generally, they may correspond to a flow which acts on the metric as a diffeomorphism, so that the action is scale-independent up to field redefinitions. This requires

$$\beta_{ij} = \pounds_v\, g_{ij}, \tag{1.12}$$

where v is a vector field on $\mathcal{M}$ and $\pounds_v$ denotes the Lie derivative [325] along v. In the one–loop approximation, the LHS is proportional to the Ricci tensor, and the metrics which solve equation (1.12) are precisely the *Ricci solitons* [95].

As we shall see in Chapter 2, the σ–model admits a supersymmetry (SUSY) completion. The above discussion for the β–function extends to the SUSY case.

1.1.2 Symmetries, gaugings, and Killing vectors

The geometry says more. Assume our Lagrangian field theory is invariant under a continuous symmetry group G which acts on the scalar fields ϕ^i. G should be, in particular, a symmetry of the two–derivative terms in $\mathcal{L}$, Eq. (1.1), hence it should leave invariant the metric g_{ij}; that is, *G should be a subgroup of the isometry group* $\mathrm{Iso}(\mathcal{M})$ of the Riemannian manifold $\mathcal{M}$.

The corresponding infinitesimal symmetries, $\phi^i \to \phi^i + \epsilon^A K_A{}^i(\phi)$, are generated by vector fields $K_A{}^i\,\partial_i$ ($A = 1, 2 \dots, \dim G$) which satisfy the Killing condition

$$\pounds_{K_A} g_{ij} \equiv \nabla_i K_{Aj} + \nabla_j K_{Ai} = 0, \tag{1.13}$$

as well as the algebra

$$\pounds_{K_A} K_B = [K_A, K_B] = f_{AB}{}^C K_C, \tag{1.14}$$

where $f_{AB}{}^C$ are the structure constants of $\mathfrak{g}$ (the Lie algebra of G).

The existence of a non–trivial group of isometries – in particular a non–Abelian group – is a strong requirement on the geometry of $\mathcal{M}$. For instance, by the Bochner theorem, if $\mathcal{M}$ is compact and has negative Ricci curvature, it has *no* Killing vectors [163].

Gauging a subgroup of Iso($\mathcal{M}$)

One may wish to gauge a subgroup G of the isometry group $\mathrm{Iso}(\mathcal{M})$. The minimal coupling of the gauge vector fields A^A_μ to the scalars is also dictated by the geometry of $\mathcal{M}$ through the corresponding Killing vectors $K_A{}^i\,\partial_i$. To gauge the symmetry, one replaces in $\mathcal{L}$ the ordinary derivative by the covariant one:

$$\partial_\mu \phi^i \to D_\mu \phi^i := \partial_\mu \phi^i - A^A_\mu\, K_A{}^i. \tag{1.15}$$

The infinitesimal gauge transformation then reads

$$\delta\phi^i = \Lambda^A\, K_A{}^i, \tag{1.16}$$

$$\delta A^A_\mu = \partial_\mu \Lambda^A + f^A{}_{BC}\, A^B_\mu\, \Lambda^C, \tag{1.17}$$

where the parameters Λ^A are arbitrary functions in space–time. Then

$$\begin{aligned}\delta D_\mu \phi^i &= \Lambda^A (\partial_j K_A{}^i)\, D_\mu \phi^j \\ &\qquad + A^B_\mu \Lambda^C [K_B{}^j \partial_j K_C{}^i - K_C{}^j \partial_j K_B{}^i] - f_{BC}{}^A A^B_\mu \Lambda^C\, K_A{}^i \\ &= \Lambda^A (\partial_j K_A{}^i)\, D_\mu \phi^j.\end{aligned} \tag{1.18}$$

The covariance[3] of $D_\mu\phi^i$ follows from the closure of the gauge algebra, Eq. (1.14), while the invariance of the kinetic term $g_{ij}D_\mu\phi^i D^\mu\phi^j$ requires $\pounds_{(\Lambda^A K_A)}\, g_{ij} = 0$, i.e., the Killing condition (1.13). Indeed, from Eq. (1.18)

$$\delta(g_{ij}D_\mu\phi^i D^\mu\phi^j) = \Lambda^A(\nabla_i K_{A\,j} + \nabla_j K_{A\,i})D_\mu\phi^i D^\mu\phi^j. \tag{1.19}$$

We summarize what we have learned in the following statement:

General Lesson 1.1 *The physics of the* (gauged) *σ–model is controlled by the differential geometry of the target manifold $\mathcal{M}$.*

The physics is invariant under general reparametrizations of the target space in the same sense that General Relativity is invariant under reparametrizations of the space–time manifold. In General Relativity this invariance is often stated in the form of the *equivalence principle* [296]. The same principle holds for target space as well:

Corollary 1.2 (target space equivalence principle) *Any physical quantity which is local in $\mathcal{M}$ and depends only on the metric and its first derivative may be safely computed using a flat target space.*

In Section 1.3 we shall see the deep reason why the target space behaves as a physical space–time.

Exercise 1.1.1 Using Feymann graphs, compute the universal coefficient c_1 in Eq. (1.11) for the σ–model (1.7). Check its universality.

1.2 Adding fields of arbitrary spin

1.2.1 Couplings and geometric structures on $\mathcal{M}$

We have seen in Section 1.1.2 that the scalars' couplings to gauge vectors are specified by a set of vector fields K_A on the manifold $\mathcal{M}$ which satisfy certain differential–geometric constraints. This is a first example of a general pattern: *all* the couplings in a Lagrangian may be identified with suitable differential–geometric structures on the scalar manifold $\mathcal{M}$. To make our point, we work out the details of a specific example, in which only scalars and spin–1/2 fermions

[3] Note that $D_\mu\phi^i$ transforms under an infinitesimal gauge transformation of parameter Λ^A as the differentials $d\phi^i \in \Lambda^1(\mathcal{M})$ under the infinitesimal diffeomorphism generated by the vector field $\Lambda^A\, K_A{}^i\partial_i$. Indeed,

$$\pounds_{\Lambda^A K_A} d\phi^i \equiv \big(d\, i_{\Lambda^A K_A} + i_{\Lambda^A K_A}\, d\big)d\phi^i = d(\Lambda^A\, K_A{}^i) \equiv \Lambda^A(\partial_j K_A{}^i)d\phi^j.$$

are present. The reader can easily convince herself that the arguments are pretty general, and work, *mutatis mutandis,* for fields of arbitrary (finite) spin.

General 2D *model with fermions.*

Let us consider the general theory with scalars ϕ^i, and fermions ψ^a, where $i = 1, 2, \dots, n$, and $a = 1, 2, \dots, m$. We choose $D = 2$, which is the number of dimensions in which these models have the more interesting applications (as world-sheet theories of some superstring [200]) and in which they make sense quantum mechanically. The arguments, however, are manifestly dimension–independent (apart from questions of existence as quantum field theories).

Limiting ourselves to power–counting renormalizable theories, the most general Lagrangian is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j + b_{ij}(\phi)\, \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + V(\phi) \\ &+ i h_{ab}(\phi)\, \bar{\psi}^a \gamma^\mu \partial_\mu \psi^b + i \tilde{h}_{ab}(\phi)\, \bar{\psi}^a \gamma_3 \gamma^\mu \partial_\mu \psi^b \\ &+ k_{abi}(\phi)\, \bar{\psi}^a \gamma^\mu \psi^b \partial_\mu \phi^i + \tilde{k}_{abi}(\phi)\, \bar{\psi}^a \gamma^\mu \gamma_3 \psi^b \partial_\mu \phi^i \\ &+ y_{ab}(\phi)\, \bar{\psi}^a \psi^b + \tilde{y}_{ab}(\phi)\, \bar{\psi}^a \gamma_3 \psi^b \\ &+ s_{abcd}(\phi)\, \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d + \cdots\end{aligned} \tag{1.20}$$

where the couplings

$$\begin{gathered} g_{ij}(\phi),\ b_{ij}(\phi),\ V(\phi),\ h_{ab}(\phi),\ \tilde{h}_{ab}(\phi),\ k_{abi}(\phi), \\ \tilde{k}_{abi}(\phi),\ y_{ab}(\phi),\ \tilde{y}_{ab}(\phi),\ s_{abcd}(\phi),\ \dots \end{gathered} \tag{1.21}$$

are arbitrary functions of the scalar fields ϕ^i. Each term in the Lagrangian (1.20) may be interpreted as a geometric structure on $\mathcal{M}$. We already know that $g_{ij}(\phi)$ is a Riemannian metric. The coupling $b_{ij}(\phi)$ is antisymmetric in the indices i, j and hence can be seen as a differential 2–form $b = \frac{1}{2} b_{ij}(\phi)\, d\phi^i \wedge d\phi^j$ on $\mathcal{M}$. The value of this contribution to the action S for a given field configuration $\Phi : \Sigma \to \mathcal{M}$ is given by the very geometrical (in fact functorial) formula

$$\int_\Sigma \Phi^* b. \tag{1.22}$$

This, in particular, implies that – up to space–time boundary phenomena – the physics should be invariant under the *target space* gauge invariance

$$b \to b + d\xi \tag{1.23}$$

for all 1–forms ξ on $\mathcal{M}$. Indeed, under this variation of the $b_{ij}(\phi)$ coupling, the action changes as

$$S \to S + \int_\Sigma \Phi^* d\xi = S + \int_\Sigma d\Phi^*\xi = S + \int_{\partial\Sigma} \Phi^*\xi, \tag{1.24}$$

so $b \to b + d\xi$ is a symmetry whenever we are allowed to ignore boundary terms in the action, e.g. if the space–time Σ is closed. In such a situation, the physics should depend only on the gauge–invariant field–strength 3–form of b, $H = db$, as well as on the harmonic projection of b. In particular, when $H = 0$ – i.e., b is closed – the physics depends only on the cohomology class of b, and the coupling $\int \Phi^* b$ is purely topological: it measures the class $[\Phi^* b]$ as a multiple of the fundamental class of Σ; then $\int \Phi^* b$ does not change under continuous deformations of the map Φ.

The next coupling, $V(\phi)$, is easy. It is just a scalar field on $\mathcal{M}$. In many situations $V(\phi)$ is required to satisfy further geometric conditions. For instance, to gauge the isometry associated to a Killing vector K, we must have

$$\pounds_K V = 0, \tag{1.25}$$

which is the geometric statement of gauge invariance. Other geometric structures related to the scalar potential will be presented in Chapters 6, 7, 8.

To discuss the other couplings, we change notation and use Majorana–Weyl fermions, $\psi^a_\pm = \pm\gamma_3\psi^a_\pm$, which are the minimal spinors in 2D. Writing

$$h^\pm_{ab} = h_{ab} \pm \tilde{h}_{ab}, \tag{1.26}$$

the second line of Eq. (1.20) reads

$$i\, h^+_{ab}\, \psi^a_+ \partial_- \psi^b_+ + (+ \leftrightarrow -), \tag{1.27}$$

where, without loss of generality, we may assume the matrices $h^\pm_{ab}$ to be symmetric, since their antisymmetric part gives, up to a total derivative, terms of the form $\psi^a_+ \psi^b_+\, \partial_- h^+_{ab}$ which contribute to the third line of Eq. (1.20). The Hermitian conjugate of (1.27) is

$$-i(h^+_{ab})^* (\partial_- \psi^b_+) \psi^a_+ + (+ \leftrightarrow -). \tag{1.28}$$

Unitarity requires $h^\pm_{ab}$ to be positive–definite *real* symmetric matrices. The geometric interpretation of this last condition is obvious: the chiral fermions $\psi^a_\pm$ are *sections of vector bundles*[4] over Σ, which are the pull–backs $\Phi^*\mathcal{V}_\pm$ of real vector

[4] For the moment we take Σ to be flat Minkowski/Euclidean space. For the general case, see General Lesson 1.3 below.