

## Contents

	<i>Preface</i>	<i>page</i> xvii
	<i>Guide</i>	xx
<b>1</b>	<b>Preliminaries</b>	1
	1.1 Lebesgue measure: an outline	1
	1.1.1 Carathéodory's lemma	1
	1.1.2 Measurable sets	2
	1.1.3 The integral	4
	1.2 Probabilities and expectations	5
	1.2.1 Reduction to Lebesgue measure	5
	1.2.2 Expectation	6
	1.2.3 Independence	7
	1.3 Conditional probabilities and expectations	9
	1.3.1 Signed measures	9
	1.3.2 Radon–Nikodym	10
	1.3.3 Conditional probabilities and expectations	11
	1.3.4 Games, fair and otherwise	15
	1.4 Rademacher functions and Wiener's trick	19
	1.5 Stirling's approximation	22
	1.5.1 The poor man's Stirling	22
	1.5.2 Kelvin's method	24
	1.6 Fast Fourier: series and integrals	25
	1.6.1 Fourier series	26
	1.6.2 Fourier integral	28
	1.6.3 Poisson summation	29
	1.6.4 Several dimensions	31
	1.7 Distribution functions and densities	32
	1.7.1 Compactness	32

	1.7.2	Convolution	33
	1.7.3	Fourier transforms	34
	1.7.4	Laplace transform	35
	1.7.5	Some special densities	35
	1.8	More probability: some useful tools illustrated	36
	1.8.1	Chebyshev's Inequality	36
	1.8.2	Mills's Ratio	37
	1.8.3	Doob's Inequality	37
	1.8.4	Kolmogorov's 01 Law	38
	1.8.5	Borel–Cantelli Lemmas	39
<b>2</b>		<b>Bernoulli Trials</b>	42
	2.1	The law of large numbers (LLN)	43
	2.1.1	The weak law	43
	2.1.2	The individual or strong law	43
	2.1.3	Application of the weak law: polynomial approximation	44
	2.1.4	Application of the strong law: empirical distributions	45
	2.2	The Gaussian approximation (CLT)	48
	2.3	The law of iterated logarithm	51
	2.4	Large deviations	54
	2.4.1	Cramér's estimate	54
	2.4.2	Empirical distributions	55
	2.4.3	The descent	56
	2.4.4	Legendre–Fenchel duality	58
	2.4.5	General variables	58
<b>3</b>		<b>The Standard Random Walk</b>	60
	3.1	The Markov property	61
	3.1.1	The simple Markov property	61
	3.1.2	The strict Markov property	62
	3.2	Passage times	64
	3.2.1	A better way: the reflection principle of D. André.	66
	3.2.2	Yet another take: stopping	68
	3.2.3	Passage to a distance	68
	3.2.4	Two-sided passage: the Gambler's Ruin	69
	3.3	Loops	71
	3.3.1	Equidistribution	71
	3.3.2	The actual number of visits	73

<i>Contents</i>		ix
	3.3.3 Long runs	74
3.4	The arcsine law	76
	3.4.1 The conditional arcsine law	81
	3.4.2 Drift and the conventional wisdom	82
3.5	Volume	83
<b>4</b>	<b>The Standard Random Walk in Higher Dimensions</b>	<b>87</b>
4.1	What RW(2) and RW(3) do as $n \uparrow \infty$	87
4.2	How RW(3) escapes to $\infty$	91
	4.2.1 Speed	91
	4.2.2 Direction	92
4.3	Gauss–Landen, Pólya and RW(2)	94
4.4	RW(2): loops and occupation numbers	97
	4.4.1 Duration of a large number of loops	99
	4.4.2 Long runs	101
4.5	RW(2): a hitting distribution	101
4.6	RW(2): volume	104
4.7	RW(3): hitting probabilities	105
	4.7.1 The meaning of $G$	106
	4.7.2 Comparison with $\mathbb{R}^3$	107
	4.7.3 Electrostatics	108
	4.7.4 Back to $\mathbb{Z}^3$	108
	4.7.5 Energy and capacity in $\mathbb{R}^3$	109
	4.7.6 Finale in $\mathbb{Z}^3$	110
	4.7.7 Grounding	111
	4.7.8 Harmonic functions	111
4.8	RW(3): volume	112
4.9	Non-negative harmonic functions	115
	4.9.1 Standard walks: first pass	115
	4.9.2 Standard walks: second pass	118
	4.9.3 Variants of RW(1)	120
	4.9.4 Space-time walks, mostly in dimension 1	126
<b>5</b>	<b>LLN, CLT, Iterated Log, and Arcsine in General</b>	<b>132</b>
5.1	LLN	132
	5.1.1 Another way	133
	5.1.2 Kolmogorov’s 1933 proof	133
	5.1.3 Doob’s proof	134
5.2	Kolmogorov–Smirnov statistics	135
5.3	CLT in general	139
	5.3.1 Conventional proof	140

	5.3.2	Second proof	141
	5.3.3	Errors	143
5.4		The local limit	144
5.5		Figures of merit	146
	5.5.1	Gibbs's lemma and entropy	146
	5.5.2	Fisher's information	148
	5.5.3	Log Sobolev	148
5.6		Gauss is prime	149
5.7		The general iterated log	152
5.8		Sparre-Andersen's combinatorial method	152
	5.8.1	Sparre-Andersen's combinatorial lemma	152
	5.8.2	Application to random walk	154
	5.8.3	Spitzer's identity	155
5.9		CLT in dimensions 2 or more	157
	5.9.1	Gaussian variables	157
	5.9.2	Gauss and independence	158
	5.9.3	CLT itself	159
	5.9.4	Maxwell's distribution	160
5.10		Measure in dimension $+\infty$	162
	5.10.1	A better way	163
	5.10.2	Curvature	164
	5.10.3	A more delicate description	165
5.11		Prime numbers	166
<b>6</b>		<b>Brownian Motion</b>	170
	6.1	Preview	171
	6.2	Direct construction of BM(1)	175
	6.2.1	P. Lévy's construction	175
	6.2.2	Wiener's construction	178
	6.3	Markov property and passage times	179
	6.3.1	The simple Markov property	180
	6.3.2	The strict Markov property	180
	6.3.3	Passage times	181
	6.3.4	Two-sided passage or the Gambler's Ruin	183
	6.4	The invariance principle	184
	6.4.1	Proof	185
	6.4.2	Reprise	187
	6.5	Volume RW(1) (reprise)	188
	6.6	Arcsine (reprise)	191
	6.6.1	Feynman–Kac (FK)	192

<i>Contents</i>		xi
6.6.2	Proof by Brownian paths	194
6.6.3	Still another way by Brownian paths	196
6.7	Skorokhod embedding	199
6.7.1	CLT	200
6.7.2	The iterated log	201
6.8	Kolmogorov–Smirnov (reprise)	202
6.8.1	Tied Brownian motion	203
6.8.2	Tied Poisson walk	204
6.8.3	Evaluations	205
6.9	Itô's lemma	206
6.9.1	Brownian integrals and differentials	207
6.9.2	An example	207
6.9.3	Itô's lemma	208
6.9.4	Robert Brown and Einstein	213
6.10	Brownian motion in dimensions $\geq 2$	214
6.10.1	Itô's lemma	214
6.10.2	BM(2): some details	215
6.10.3	BM(3) and how it goes to $\infty$	217
6.11	$S^\infty(\sqrt{\infty})$ revisited	221
6.11.1	div, grad, and all that	221
6.11.2	Hermite and polynomial chaos	223
6.11.3	The Brownian format	224
6.11.4	Back to $\Delta$	226
6.11.5	Drift and Jacobian	228
<b>7</b>	<b>Markov Chains</b>	<b>231</b>
7.1	Set-up and the Markov property	232
7.2	The invariant distribution	233
7.2.1	Geometrical proof	233
7.2.2	Analytical proof	235
7.2.3	Probabilistic proof	235
7.3	LLN for chains	237
7.3.1	LLN improved	238
7.3.2	Mixing	240
7.3.3	McMillan's theorem	240
7.4	CLT for chains	241
7.4.1	Kubo's formula	243
7.4.2	CLT improved	244
7.5	Real time	244
7.5.1	The Markov property	245

	7.5.2	Loops and the invariant distribution	246
7.6		The standard Poisson process	248
7.7		Large deviations	250
	7.7.1	Setup and simplest examples of the main result	250
	7.7.2	Preliminaries about $I$	252
	7.7.3	Proof of the main result	255
	7.7.4	Legendre duality	257
<b>8</b>		<b>The Ergodic Theorem</b>	<b>260</b>
	8.1	Hamiltonian mechanics	260
	8.2	Gibbs, Birkhoff, and the statistical method	262
	8.2.1	Gibbs's canonical ensemble	262
	8.2.2	Time averages	263
	8.2.3	H. Weyl's example	264
	8.3	A more general set-up	265
	8.3.1	Metric transitivity and mixing	265
	8.3.2	Poincaré recurrence	267
	8.4	Riesz's lemma and Garsia's trick	268
	8.5	Continued fractions	270
	8.5.1	The set-up	270
	8.5.2	Birkhoff applied	272
	8.5.3	Proof of metric transitivity	273
	8.5.4	Mixing	274
	8.5.5	Information rate (McMillan's theorem)	275
	8.6	Geodesic flow	278
	8.6.1	Sphere	278
	8.6.2	Plane	279
	8.6.3	Poincaré's half-plane	279
	8.6.4	$\mathbb{H}^2/\Gamma$ : the circle bundle	284
	8.6.5	How continued fractions enter	285
	8.6.6	CLT	288
	8.6.7	Back to $\mathbb{R}^2/\mathbb{Z}^2$	288
	8.6.8	Why $\mathbb{H}^2/\Gamma$ is better	289

<i>Contents</i>		xiii
<b>9</b>	<b>Communication over a Noisy Channel</b>	290
9.1	Information/Uncertainty/Entropy	291
9.1.1	What Boltzmann said	294
9.1.2	Information as a guide to gambling	296
9.1.3	A dishonest coin	297
9.1.4	Relative entropy	298
9.2	Noiseless coding	299
9.3	The source	300
9.3.1	The rate	301
9.3.2	McMillan's theorem (reprise)	302
9.4	The noisy channel: capacity	305
9.4.1	Simplest example	306
9.5	The noisy channel: coding	308
9.6	Communication when $H > C$	309
9.7	Communication when $H < C$	310
9.8	The binary symmetric channel	313
9.8.1	Shannon's idea for $H < C$	313
9.8.2	Garbage out	316
<b>10</b>	<b>Equilibrium Statistical Mechanics</b>	317
10.1	What Gibbs said	317
10.1.1	Phase space and energy	317
10.1.2	The microcanonical ensemble	318
10.1.3	Heat bath and the canonical ensemble	319
10.1.4	Large volume	320
10.1.5	Thermodynamics: free energy	321
10.1.6	Thermodynamics: pressure	323
10.2	Two simple examples	326
10.2.1	Ideal gas	326
10.2.2	Hard balls	326
10.3	Van der Waals' gas law: dimension 1	328
10.4	Van der Waals: dimension 3	331
10.4.1	$Z$ bounded below	331
10.4.2	Scaling and the van der Waals limit	333
10.4.3	Finishing the proof	333
10.5	The Ising model	334
10.5.1	Overview	334
10.5.2	Dimension 1	336
10.5.3	Dimension 2	337
10.6	Existence of $\mathfrak{J}$	338

10.7	Magnetization per spin: dimension 2	341
10.7.1	Shape of $\mathbf{m}$ , $T$ fixed	341
10.7.2	Shape of $\mathbf{m}$ , $K$ fixed	342
10.8	Change of phase: dimension 2	345
10.8.1	High temperature	345
10.8.2	Low temperature	348
10.9	Duality and the critical temperature	350
<b>11</b>	<b>Statistical Mechanics Out of Equilibrium</b>	<b>352</b>
11.1	What Boltzmann said and what came after	354
11.2	The two-speed gas: chaos and the law of large numbers	363
11.2.1	The empirical distribution	364
11.2.2	Chaos and the law of large numbers	365
11.2.3	Why chaos propagates	367
11.3	The two-speed gas: fluctuations	369
11.4	More about Boltzmann's equation	371
11.5	The two-speed gas with streaming	374
11.5.1	Solving Boltzmann	375
11.5.2	Carleman's gas	376
11.5.3	The surprising equation	377
11.5.4	Velocity and displacement	380
11.6	Chapman–Enskog–Hilbert	381
11.6.1	First pass	382
11.6.2	Second pass	383
11.6.3	Making better sense of all that	384
11.6.4	Focusing	385
11.7	Kac's gas	385
11.7.1	Boltzmann's equation and Wild's sum	386
11.7.2	Entropy and the tendency to equilibrium	390
11.7.3	Fisher's information	391
11.7.4	CLT: Trotter's method	393
11.7.5	CLT: Grünbaum's method	395
11.7.6	A tagged molecule	397
11.7.7	CLT: $S^\infty(\sqrt{\infty})$ revisited	400

*Contents*

xv

<b>12</b>	<b>Random Matrices</b>	402
12.1	The Gaussian orthogonal ensemble (GOE)	402
12.2	Why a semi-circle?	404
	12.2.1 Reduction to spec $\mathbf{x}$	404
	12.2.2 Steepest descent	405
12.3	The semi-circle: a hands-on proof	408
	12.3.1 Wiener's recipe	409
	12.3.2 Traces	409
	12.3.3 Samples	410
	12.3.4 A better way	411
	12.3.5 Leading order	412
	12.3.6 Convergence of traces	413
	12.3.7 The semi-circle	414
12.4	Dyson's Coulomb gas	415
	12.4.1 $2 \times 2$ hands-on	415
	12.4.2 $n \times n$ in general	416
	12.4.3 Coda	418
12.5	Brownian motion without crossing	419
	12.5.1 Crossing times	419
	12.5.2 Connection to spec $\mathbf{x}$	420
12.6	The Gaussian unitary ensemble	422
	12.6.1 Reduction to spec $\mathbf{x}$	422
	12.6.2 Scaling and the semi-circle	423
	12.6.3 Dyson's gas	423
12.7	How to compute	424
	12.7.1 Andréief's lemma	424
	12.7.2 Application to spec $\mathbf{x}$ : first pass	425
	12.7.3 Hermite polynomials	426
	12.7.4 Application to spec $\mathbf{x}$ : second pass	427
	12.7.5 Fredholm determinants	428
	12.7.6 Application to spec $\mathbf{x}$ : third pass	430
12.8	In the bulk	431
	12.8.1 A single gap	432
	12.8.2 Wigner's surmise	433
12.9	The ODE	434
12.10	The tail	439
	12.10.1 Wigner's surmise (corrected)	442
12.11	At the edge	443
12.12	Coda	444
	12.12.1 Some history old and new	444

12.12.2 What's happening	445
12.12.3 Riemann and the prime numbers	445
<i>Bibliography</i>	447
<i>Index</i>	458