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Part I

Derived functors and homotopy (co)limits

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1

All concepts are Kan extensions

Given a pair of functors $K: \mathcal{C} \to \mathcal{D}$, $F: \mathcal{C} \to \mathcal{E}$, it may or may not be possible to extend F along K. Obstructions can take several forms: two arrows in $\mathcal{C}$ with distinct images in $\mathcal{E}$ might be identified in $\mathcal{D}$, or two objects might have empty hom-sets in $\mathcal{C}$ and $\mathcal{E}$ but not in $\mathcal{D}$. In general, it is more reasonable to ask for a best approximation to an extension taking the form of a universal natural transformation pointing either from or to F. The resulting categorical notion, quite simple to define, is surprisingly ubiquitous throughout mathematics, as we shall soon discover.

1.1 Kan extensions

Definition 1.1.1 Given functors $F: \mathcal{C} \to \mathcal{E}$, $K: \mathcal{C} \to \mathcal{D}$, a **left Kan extension** of F along K is a functor $\mathrm{Lan}_K F: \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta: F \Rightarrow \mathrm{Lan}_K F \cdot K$ such that for any other such pair $(G: \mathcal{D} \to \mathcal{E}, \gamma: F \Rightarrow GK)$, γ factors uniquely through η, as illustrated:[1]

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\exists!\Downarrow"', near end] &
\end{tikzcd}$$

Dually, a **right Kan extension** of F along K is a functor $\mathrm{Ran}_K F: \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\epsilon: \mathrm{Ran}_K F \cdot K \Rightarrow F$ such that for any

[1] Writing α for the natural transformation $\mathrm{Lan}_K F \Rightarrow G$, the right-hand **pasting diagrams** express the equality $\gamma = \alpha_K \cdot \eta$, i.e., that γ factors as $F \xRightarrow{\eta} \mathrm{Lan}_K F \cdot K \xRightarrow{\alpha_K} GK$.

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$(G\colon \mathcal{D} \to \mathcal{E}, \delta\colon GK \Rightarrow F)$, δ factors uniquely through ϵ, as illustrated:

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, dashed, "\mathrm{Ran}_K F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\delta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\epsilon"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\mathrm{Ran}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\exists!\Nwarrow"'] &
\end{tikzcd}
$$

Remark 1.1.2 This definition makes sense in any 2-category, but for simplicity, this discussion is relegated to the 2-category <u>**Cat**</u> of categories, functors, and natural transformations.

The intuition is clearest when the functor K of Definition 1.1.1 is an inclusion; assuming certain (co)limits exist, when K is fully faithful, the left and right Kan extensions do in fact extend the functor F along K; see 1.4.5. However in general, this need not be the case:

Exercise 1.1.3 Construct a toy example to illustrate that if F factors through K along some functor H, it is not necessarily the case that $(H, 1_F)$ is the left Kan extension of F along K.

Remark 1.1.4 In unenriched category theory, a **universal property** is encoded as a representation for an appropriate **Set**-valued functor. A left Kan extension of $F\colon \mathcal{C} \to \mathcal{E}$ along $K\colon \mathcal{C} \to \mathcal{D}$ is a representation for the functor

$$\mathcal{E}^{\mathcal{C}}(F, - \circ K)\colon \mathcal{E}^{\mathcal{D}} \to \mathbf{Set}$$

that sends a functor $\mathcal{D} \to \mathcal{E}$ to the set of natural transformations from F to its restriction along K. By the Yoneda lemma, any pair (G, γ) as in Definition 1.1.1 defines a natural transformation

$$\mathcal{E}^{\mathcal{D}}(G, -) \overset{\gamma}{\Longrightarrow} \mathcal{E}^{\mathcal{C}}(F, - \circ K).$$

The universal property of the pair $(\mathrm{Lan}_K F, \eta)$ is equivalent to the assertion that the corresponding map

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, -) \overset{\eta}{\Longrightarrow} \mathcal{E}^{\mathcal{C}}(F, - \circ K)$$

is a natural isomorphism, that is, that $(\mathrm{Lan}_K F, \eta)$ represents this functor.

Extending this discussion, it follows that if, for fixed K, the left and right Kan extensions of any functor $\mathcal{C} \to \mathcal{E}$ exist, then these define left and right

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adjoints to the precomposition functor $K^*\colon \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$:

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, G) \cong \mathcal{E}^{\mathcal{C}}(F, GK) \quad \begin{tikzcd} \mathcal{E}^{\mathcal{C}} \arrow[r, bend left, "\mathrm{Lan}_K"] \arrow[r, bend right, "\mathrm{Ran}_K"'] & \mathcal{E}^{\mathcal{D}} \arrow[l, "K^*"'] \end{tikzcd} \quad \mathcal{E}^{\mathcal{C}}(GK, F) \cong \mathcal{E}^{\mathcal{D}}(G, \mathrm{Ran}_K F) \tag{1.1.5}$$

The 2-cells η are the components of the unit for $\mathrm{Lan}_K \dashv K^*$, and the 2-cells ϵ are the components of the counit for $K^* \dashv \mathrm{Ran}_K$. The universal properties of Definition 1.1.1 are precisely those required to define the value at a particular object $F \in \mathcal{E}^{\mathcal{C}}$ of a left and right adjoint to a specified functor, in this case K^*.

Conversely, by uniqueness of adjoints, the objects in the image of any left or right adjoint to a precomposition functor are Kan extensions. This observation leads to several immediate examples.

Example 1.1.6 A small category with a single object and only invertible arrows is precisely a (discrete) group. The objects of the functor category $\mathbf{Vect}_k^G$ are G-representations over a fixed field k; arrows are G-equivariant linear maps. If H is a subgroup of G, restriction $\mathbf{Vect}_k^G \to \mathbf{Vect}_k^H$ of a G-representation to an H-representation is simply precomposition by the inclusion functor $i\colon H \hookrightarrow G$. This functor has a left adjoint, induction, which is left Kan extension along i. The right adjoint, coinduction, is right Kan extension along i:

$$\begin{tikzcd} \mathbf{Vect}_k^G \arrow[r, "\mathrm{res}"] & \mathbf{Vect}_k^H \arrow[l, bend right, "\mathrm{ind}_H^G"'] \arrow[l, bend left, "\mathrm{coin}_H^G"] \end{tikzcd} \tag{1.1.7}$$

The reader unfamiliar with the construction of induced representations need not remain in suspense for very long; see Theorem 1.2.1 and Example 1.2.9. Similar remarks apply for G-sets, G-spaces, based G-spaces, or indeed G-objects in any category – although in the general case, these adjoints might not exist.

Remark 1.1.8 This example can be enriched (cf. 7.6.9): extension of scalars, taking an R-module M to the S-module $M \otimes_R S$, is the **Ab**-enriched left Kan extension along an **Ab**-functor $R \to S$ between one-object **Ab**-categories, more commonly called a ring homomorphism.

Example 1.1.9 Let $\mathbb{\Delta}$ be the category of finite non-empty ordinals $[0], [1], \ldots$ and order-preserving maps. **Set**-valued presheaves on $\mathbb{\Delta}$ are called **simplicial sets**. Write $\mathbb{\Delta}_{\leq n}$ for the full subcategory on the objects $[0], \ldots, [n]$. Restriction

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along the inclusion functor $i_n : \mathbb{\Delta}_{\leq n} \hookrightarrow \mathbb{\Delta}$ is called n-truncation. This functor has both left and right Kan extensions:

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{\Delta}^{op}} \arrow[r, "i_n^*"] & \mathbf{Set}^{\mathbb{\Delta}^{op}_{\leq n}} \arrow[l, bend right, "\mathrm{Lan}_{i_n}"'] \arrow[l, bend left, "\mathrm{Ran}_{i_n}"] \end{tikzcd}$$

The composite comonad on $\mathbf{Set}^{\mathbb{\Delta}^{op}}$ is sk_n, the functor that maps a simplicial set to its n-**skeleton**. The composite monad on $\mathbf{Set}^{\mathbb{\Delta}^{op}}$ is cosk_n, the functor that maps a simplicial set to its n-**coskeleton**. Furthermore, sk_n is left adjoint to cosk_n, as is the case for any comonad and monad arising in this way.

Example 1.1.10 The category $\mathbb{\Delta}$ is a full subcategory containing all but the initial object $[-1]$ of the category $\mathbb{\Delta}_+$ of finite ordinals and order-preserving maps. Presheaves on $\mathbb{\Delta}_+$ are called **augmented simplicial sets**. Left Kan extension defines a left adjoint to restriction,

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{\Delta}_+^{op}} \arrow[r, "\mathrm{res}"] & \mathbf{Set}^{\mathbb{\Delta}^{op}} \arrow[l, bend right, "\pi_0"'] \arrow[l, bend left, "\mathrm{triv}"] \end{tikzcd}$$

that augments a simplicial set X with its set $\pi_0 X$ of path components. Right Kan extension assigns a simplicial set the trivial augmentation built from the one-point set.

A final broad class of examples has a rather different flavor.

Example 1.1.11 In good situations, the composite of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories equipped with subcategories of "weak equivalences" and the localization functor $\mathcal{D} \to \mathrm{Ho}\mathcal{D}$ admits a right or left Kan extension along the localization functor $\mathcal{C} \to \mathrm{Ho}\mathcal{C}$, called the **total left derived functor** or **total right derived functor**, respectively. This is the subject of Chapter 2.

1.2 A formula

Importantly, if the target category $\mathcal{E}$ has certain limits and colimits, then right and left Kan extensions for any pair of functors exist and furthermore can be computed by a particular (co)limit formula. Recall that a category is **small** if it

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has a mere set of morphisms and **locally small** if it has a mere set of morphisms between any fixed pair of objects.

Theorem 1.2.1 ([50, X.4.1–2]) *When $\mathcal{C}$ is small, $\mathcal{D}$ is locally small, and $\mathcal{E}$ is cocomplete, the left Kan extension of any functor $F: \mathcal{C} \to \mathcal{E}$ along any functor $K: \mathcal{C} \to \mathcal{D}$ is computed at $d \in \mathcal{D}$ by the colimit*

$$\mathrm{Lan}_K F(d) = \int^{c \in \mathcal{C}} \mathcal{D}(Kc, d) \cdot Fc \tag{1.2.2}$$

and in particular necessarily exists.

Some explanation is in order. The "$\cdot$" is called a **copower** or a **tensor**: if S is a set and $e \in \mathcal{E}$, then $S \cdot e$ is the S-indexed coproduct of copies of e. Assuming these coproducts exist in $\mathcal{E}$, the copower defines a bifunctor $\mathbf{Set} \times \mathcal{E} \to \mathcal{E}$.

The integral $\int^{\mathcal{C}}$, called a **coend**, is the colimit of a particular diagram constructed from a functor that is both covariant and contravariant in $\mathcal{C}$. Given $H: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$, the coend $\int^{\mathcal{C}} H$ is an object of $\mathcal{E}$ equipped with arrows $H(c, c) \to \int^{\mathcal{C}} H$ for each $c \in \mathcal{C}$ that are collectively universal with the property that the diagram

$$\begin{tikzcd} H(c', c) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(c', c') \arrow[d] \\ H(c, c) \arrow[r] & \int^{\mathcal{C}} H \end{tikzcd} \tag{1.2.3}$$

commutes for each $f: c \to c'$ in $\mathcal{C}$. Equivalently, $\int^{\mathcal{C}} H$ is the coequalizer of the diagram

$$\coprod_{f \in \mathrm{arr}\,\mathcal{C}} H(\mathrm{cod}\, f, \mathrm{dom}\, f) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in \mathrm{ob}\,\mathcal{C}} H(c, c) \dashrightarrow \int^{\mathcal{C}} H \tag{1.2.4}$$

Remark 1.2.5 If $H: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ is constant in the first variable, that is, if H is a functor $\mathcal{C} \to \mathcal{E}$, then the coequalizer (1.2.4) defines the usual colimit of H.

Remark 1.2.6 Assuming these colimits exist, the coend (1.2.2) is isomorphic to the colimit of the composite $K/d \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{E}$ of F with a certain forgetful functor. The domain of U is the **slice category**, a special kind of **comma category** whose objects are pairs $(c \in \mathcal{C}, Kc \to d \in \mathcal{D})$ and whose morphisms are arrows in $\mathcal{C}$ that make the obvious triangle in $\mathcal{D}$ commute. Both formulas

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encode a particular **weighted colimit** of F in a sense that is made precise in Chapter 7. In particular, we prove that these formulas agree in 7.1.11.

Exercise 1.2.7 Let $\mathcal{C}$ be a small category, and write $\mathcal{C}^{\triangleright}$ for the category obtained by adjoining a terminal object to $\mathcal{C}$. Give three proofs that a left Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along the natural inclusion $\mathcal{C} \to \mathcal{C}^{\triangleright}$ defines a colimit cone under F: one using the defining universal property, one using Theorem 1.2.1, and one using the formula of 1.2.6.

Dually, the **power** or **cotensor** e^S of $e \in \mathcal{E}$ by a set S is the S-indexed product of copies of e, defining a bifunctor $\mathbf{Set}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}$ that is contravariant in the indexing set. For $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$, an **end** $\int_{\mathcal{C}} H$ is an object in $\mathcal{E}$ together with morphisms satisfying diagrams dual to (1.2.3) and universal with this property.

Exercise 1.2.8 Let $F, G : \mathcal{C} \rightrightarrows \mathcal{E}$, with $\mathcal{C}$ small and $\mathcal{E}$ locally small. Show that the end over $\mathcal{C}$ of the bifunctor $\mathcal{E}(F-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is the set of natural transformations from F to G.

Example 1.2.9 Let us return to Example 1.1.6. In the category $\mathbf{Vect}_k$, finite products and finite coproducts coincide: these are just direct sums of vector spaces. If V is an H-representation and H is a finite index subgroup of G, then the end and coend formulas of Theorem 1.2.1 and its dual both produce the direct sum of copies of V indexed by left cosets of H in G. Thus, for finite index subgroups, the left and right adjoints of (1.1.7) are the same; that is, induction from a finite index subgroup is both left and right adjoint to restriction.

Example 1.2.10 We can use Theorem 1.2.1 to understand the functors sk_n and cosk_n of Example 1.1.9. If $m > n$ and $k \le n$, each map in $\mathbb{\Delta}^{\mathrm{op}}([k], [m]) = \mathbb{\Delta}([m], [k])$ factors uniquely as a non-identity epimorphism followed by a monomorphism.[2] It follows that every simplex in $\mathrm{sk}_n X$ above dimension m is degenerate; indeed, $\mathrm{sk}_n X$ is obtained from the n-truncation of X by freely adding back the necessary degenerate simplices.

Now we use the adjunction $\mathrm{sk}_n \dashv \mathrm{cosk}_n$ to build some intuition for the n-coskeleton. Suppose $X \cong \mathrm{cosk}_n X$. By adjunction, an $(n+1)$-simplex corresponds to a map $\mathrm{sk}_n \Delta^{n+1} = \partial \Delta^{n+1} \to X$. In words, each $(n+1)$-sphere in an n-coskeletal simplicial set has a unique filler. Indeed, any m-sphere in an n-coskeletal simplicial set, with $m > n$, has a unique filler. More precisely, an m-simplex is uniquely determined by the data of its faces of dimension n and below.

Exercise 1.2.11 Directed graphs are functors from the category with two objects E, V and a pair of maps $s, t : E \rightrightarrows V$ to $\mathbf{Set}$. A natural transformation between two such functors is a graph morphism. The forgetful functor

[2] This is the content of the Eilenberg–Zilber lemma [28, II.3.1, pp. 26–27]; cf. Lemma 14.3.7.

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DirGph $\to$ **Set** that maps a graph to its set of vertices is given by restricting along the functor from the terminal category $\mathbb{1}$ that picks out the object V. Use Theorem 1.2.1 to compute left and right adjoints to this forgetful functor.

1.3 Pointwise Kan extensions

A functor $L\colon \mathcal{E} \to \mathcal{F}$ **preserves** $(\mathrm{Lan}_K F, \eta)$ if the whiskered composite $(L\mathrm{Lan}_K F, L\eta)$ is the left Kan extension of LF along K:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \arrow[r, "L"] & \mathcal{F} \\ & \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F"'] & & \end{tikzcd} \cong \begin{tikzcd} \mathcal{C} \arrow[rr, "LF"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta'"] & \mathcal{F} \\ & \mathcal{D} \arrow[ur, "\mathrm{Lan}_K LF"'] & \end{tikzcd}$$

Example 1.3.1 The forgetful functor $U\colon$ **Top** $\to$ **Set** has both left and right adjoints and hence preserves both limits and colimits. It follows from Theorem 1.2.1 that U preserves the left and right Kan extensions of Example 1.1.6.

Example 1.3.2 The forgetful functor $U\colon \mathbf{Vect}_k \to$ **Set** preserves limits but not colimits because the underlying set of a direct sum is not simply the coproduct of the underlying sets of vectors. Hence it follows from 1.2.1 and 1.1.6 that the underlying set of a G-representation induced from an H-representation is not equal to the G-set induced from the underlying H-set.

Even when we cannot appeal to the formula presented in 1.2.1 adjoint functors preserve compatibly handed Kan extensions:

Lemma 1.3.3 *Left adjoints preserve left Kan extensions.*

Proof Suppose given a left Kan extension $(\mathrm{Lan}_K F, \eta)$ with codomain $\mathcal{E}$ and suppose further that $L\colon \mathcal{E} \to \mathcal{F}$ has a right adjoint R with unit ι and counit ν. Then, given $H\colon \mathcal{D} \to \mathcal{F}$, there are natural isomorphisms

$$\mathcal{F}^{\mathcal{D}}(L\mathrm{Lan}_K F, H) \cong \mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_K F, RH) \cong \mathcal{E}^{\mathcal{C}}(F, RHK) \cong \mathcal{F}^{\mathcal{C}}(LF, HK).$$

Taking $H = L\mathrm{Lan}_K F$, these isomorphisms act on the identity natural transformation, as follows:

$$1_{L\mathrm{Lan}_K F} \mapsto \iota_{\mathrm{Lan}_K F} \mapsto \iota_{\mathrm{Lan}_K F \cdot K} \cdot \eta \mapsto \nu_{L\mathrm{Lan}_K F \cdot K} \cdot L\iota_{\mathrm{Lan}_K F \cdot K} \cdot L\eta = L\eta.$$

Hence $(L\mathrm{Lan}_K F, L\eta)$ is a left Kan extension of LF along K. □

Unusually for a mathematical object defined by a universal property, generic Kan extensions are rather poorly behaved. We see specific examples of this insufficiency in Chapter 2, but for now we have to rely on expert opinion. For

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instance, Max Kelly reserves the name "Kan extension" for pairs satisfying the condition we presently introduce, calling those of our Definition 1.1.1 "weak" and writing that "our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever" [46, §4]. By the categorical community's consensus, the important Kan extensions are **pointwise** Kan extensions.

Definition 1.3.4 When $\mathcal{E}$ is locally small, a right Kan extension is a **pointwise** right Kan extension[3] if it is preserved by all representable functors $\mathcal{E}(e, -)$.

Because covariant representables preserve all limits, it is clear that if a right Kan extension is given by the formula of Theorem 1.2.1, then that Kan extension is pointwise; dually, left Kan extensions computed in this way are pointwise. The surprise is that the converse also holds. This characterization justifies the terminology: a pointwise Kan extension can be computed pointwise as a limit in $\mathcal{E}$.

Theorem 1.3.5 ([50, X.5.3]) *A right Kan extension of F along K is pointwise if and only if it can be computed by*

$$\mathrm{Ran}_K F(d) = \lim \left(d/K \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{E} \right)$$

in which case, in particular, this limit exists.

Proof If $\mathrm{Ran}_K F$ is pointwise, then by the Yoneda lemma and the defining universal property of right Kan extensions,

$$\begin{aligned} \mathcal{E}(e, \mathrm{Ran}_K F(d)) &\cong \mathbf{Set}^{\mathcal{D}}(\mathcal{D}(d, -), \mathcal{E}(e, \mathrm{Ran}_K F)) \\ &\cong \mathbf{Set}^{\mathcal{C}}(\mathcal{D}(d, K-), \mathcal{E}(e, F-)). \end{aligned}$$

The right-hand set is naturally isomorphic to the set of cones under e over the functor FU; hence this bijection exhibits $\mathrm{Ran}_K F(d)$ as the limit of FU. □

Remark 1.3.6 Most commonly, pointwise Kan extensions are found whenever the codomain category is cocomplete (for left Kan extensions) or complete (for right), but this is not the only case. In Chapter 2, we see that the most common construction of the total derived functors defined in 1.1.11 produces pointwise Kan extensions, even though homotopy categories have notoriously few limits and colimits (see Proposition 2.2.13).

[3] A functor $K\colon \mathcal{C} \to \mathcal{D}$ is equally a functor $K\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, but the process of replacing each category by its opposite reverses the direction of any natural transformations; succinctly, "op" is a 2-functor $(-)^{\mathrm{op}}\colon \underline{\mathbf{Cat}}^{\mathrm{co}} \to \underline{\mathbf{Cat}}$. A left Kan extension is **pointwise**, as we are in the process of defining, if the corresponding right Kan extension in the image of this 2-functor is pointwise.

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