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A Primer on the Dirichlet Space

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To:
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Julian and Étienne

Cambridge University Press
978-1-107-04752-5 - A Primer on the Dirichlet Space
Omar El-Fallah, Karim Kellay, Javad Mashreghi and Thomas Ransford
Frontmatter
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Preface

The three classical Hilbert spaces of holomorphic functions in the unit disk are the Hardy, Bergman and Dirichlet spaces. There are several excellent texts covering the Hardy space and the Bergman space. However, to the best of our knowledge, up to now there has been no book devoted to the Dirichlet space. When we began our respective researches into the Dirichlet space, we found ourselves handicapped by the fact that the necessary background information was scattered around the literature, sometimes contained in articles that were difficult to follow. For this reason we began to think about writing an introduction that would be suitable for researchers and graduate students seeking a solid background in the subject. The more we learned about this topic, the more we became convinced that it contains many beautiful ideas that deserve a systematic exposition.

The name Dirichlet space derives from its definition in terms of the so-called Dirichlet integral, arising in Dirichlet's method for solving Laplace's equation (sometimes called the Dirichlet principle). As far as we can determine, the first appearance of the Dirichlet space under that name dates back to two articles of Beurling and Deny in 1958 and 1959, but in fact the notion existed and had been studied at least since Beurling's thesis, which was published in 1933 and written even a little earlier. In the years that followed, Beurling and Carleson laid the foundations of the theory and, after their pioneering work, many other distinguished mathematicians made important contributions.

Why study the Dirichlet space? Here are a few reasons.

(1) The Hardy space corresponds to ℓ^2, the Hilbert space of square-summable sequences. One of the main advantages of thinking of it as a function space is that the shift operator on ℓ^2 becomes simply multiplication by z. If one is interested in weighted shifts on ℓ^2, which are very important in operator theory, then one should consider multiplication by z on a weighted function

space. The two most basic non-constant weights lead one immediately to the Dirichlet and Bergman spaces.

(2) The Dirichlet integral of a holomorphic function f has a very natural geometric interpretation. It is exactly the area of the image of f, counted according to multiplicity. Seen this way, it is obviously invariant under precomposition with every Möbius automorphism of the unit disk. It is a remarkable fact that this Möbius-invariance property characterizes the Dirichlet space among all Hilbert function spaces on the disk.

(3) The Dirichlet space is closely related to logarithmic potential theory. In particular, the notions of energy and logarithmic capacity play a prominent role in the theory. This reflects Beurling's vision of the subject, and yields interesting interactions with physics.

(4) The Dirichlet integral is the motivating example of the abstract notion of a Dirichlet form, first introduced by Beurling and Deny in the articles mentioned above. Dirichlet forms have become a fundamental tool in probability and semigroup theory (though this is not an aspect that will be developed in this book).

(5) From many points of view, the Dirichlet space is a borderline case. For example, it is very nearly an algebra, but not quite. This borderline nature makes it an interesting and challenging example of a function space. Many important questions remain unsolved, and the Dirichlet space is still an active area of research.

What is in the book? To get a quick idea, imagine being presented with a function space on the unit disk. Several standard questions naturally arise. For example:

- What can be said about the boundary behavior of functions in the space?
- Are there simple characterizations of zero sets and uniqueness sets?
- What can we say about interpolation?
- Is the space an algebra? If not, then what are the multipliers?
- How rich is the operator theory on this function space? For example, can we classify the shift-invariant subspaces? Which functions are cyclic?

In the case of the Hardy space, the answers to all these questions are well known and important. By contrast, in the Dirichlet space, some of the questions have been only partially answered, and even where the complete answers are known, they are more subtle. This is the subject of this book.

Perhaps it is also worth mentioning what is not in the book. As it is meant to be a primer, we do not pretend to give an exhaustive treatment of the subject, and certain topics, such as interpolating sequences and the corona problem, have been omitted completely (with much regret). We have decided to restrict ourselves to the classical Dirichlet space, treating other variants such as weighted Dirichlet spaces when they contribute directly to understanding the classical case.

The prerequisites are a knowledge of standard complex analysis, measure theory and functional analysis. Also, we have taken for granted a certain familiarity with Hardy spaces, the necessary background being summarized briefly in an appendix. We do however develop the notion of logarithmic capacity *ab initio*, since it turns up throughout the book.

There are exercises at the end of most of the sections, ranging from routine calculations to barely disguised theorems. We have tried our best to attribute results correctly, in notes at the end of each chapter. However, history is sometimes complicated, and we apologize if we have fallen short of our aim.

In the course of writing the book, we have benefitted from discussions with many mathematicians. In particular, we thank Alexandru Aleman, Nicola Arcozzi, Sasha Borichev, Håkan Hedenmalm, Stefan Richter, Bill Ross, Kristian Seip and Andrew Wynn. Also we thank Jérémie Rostand for his help with the illustrations. We are grateful to Roger Astley and his colleagues at Cambridge University Press for their advice and encouragement. Part of this book was written at the CIRM (Luminy), and we express our gratitude to the CIRM for its hospitality. We gratefully acknowledge the financial support of the following granting bodies: CNRST and the Hassan II Academy of Science and Technology (OE), PICS-CNRS (KK), NSERC (JM and TR) and the Canada research chairs program (TR). Of course, we owe a huge debt of gratitude to our spouses, Salma, Nathalie, Shahzad and Line, for supporting us and putting up with us while the book was being written. Last, but not least, we thank our children for constantly reminding us that there are things even more important than mathematics. We dedicate the book to them.