1

Basic notions

In this introductory chapter, we introduce the Dirichlet space. We develop its elementary properties, at the same time pointing the way to the deeper results to be treated in the subsequent chapters.

1.1 The Dirichlet space

Let us begin with the fundamental definition. In what follows, $\mathbb{D}$ denotes the open unit disk, $\mathrm{Hol}(\mathbb{D})$ is the set of all functions holomorphic in $\mathbb{D}$, and dA denotes the area Lebesgue measure on the complex plane $\mathbb{C}$.

Definition 1.1.1 Given $f \in \mathrm{Hol}(\mathbb{D})$, the *Dirichlet integral* of f is defined by

$$\mathcal{D}(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

The *Dirichlet space* $\mathcal{D}$ is the vector space of $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}(f) < \infty$.

Clearly $\mathcal{D}$ contains all the polynomials and, more generally, all functions holomorphic on $\overline{\mathbb{D}}$ such that f' is bounded on $\mathbb{D}$. We shall see that it also contains many other interesting functions.

Our first result is a formula for $\mathcal{D}(f)$ in terms of the Taylor coefficients of f.

Theorem 1.1.2 *Let $f \in \mathrm{Hol}(\mathbb{D})$, say $f(z) = \sum_{k\geq 0} a_k z^k$. Then*

$$\mathcal{D}(f) = \sum_{k\geq 1} k|a_k|^2. \tag{1.1}$$

Proof Writing the area integral in polar coordinates, we have

$$\mathcal{D}(f) = \frac{1}{\pi}\int_{\mathbb{D}} \Big|\sum_{k\geq 1} k a_k z^{k-1}\Big|^2 \, dA(z) = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \Big|\sum_{k\geq 1} k a_k r^{k-1} e^{i(k-1)\theta}\Big|^2 \, d\theta \, r \, dr.$$

By Parseval's formula, for each $r \in (0, 1)$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{k\geq 1} ka_k r^{k-1}e^{i(k-1)\theta}\Big|^2\,d\theta = \sum_{k\geq 1} k^2|a_k|^2 r^{2k-2}.$$

Hence

$$\mathcal{D}(f) = 2\int_0^1 \sum_{k\geq 1} k^2|a_k|^2 r^{2k-1}\,dr = \sum_{k\geq 1} k|a_k|^2. \qquad \square$$

From this, we derive the following simple but useful observation. Recall that the Hardy space H^2 consists of those holomorphic functions $f(z) = \sum_{k\geq 0} a_k z^k$ such that $\|f\|_{H^2}^2 := \sum_{k\geq 0}|a_k|^2 < \infty$.

Corollary 1.1.3 *The Dirichlet space $\mathcal{D}$ is contained in the Hardy space H^2.*

Proof This follows immediately from the theorem, together with the obvious fact that $\sum_{k\geq 0} k|a_k|^2 < \infty$ implies $\sum_{k\geq 0}|a_k|^2 < \infty$. $\square$

We shall make frequent use of this inclusion, exploiting the many known properties of the Hardy space H^2. A summary of the properties that we shall need can be found in Appendix A.

Notice that $\mathcal{D}$ is dense in H^2, because it contains the polynomials. As it is obviously a proper subspace of H^2, it is not closed in H^2, so it is not a Hilbert space with respect to $\|\cdot\|_{H^2}$. We now endow it with a norm, making it a Hilbert space in its own right.

For $f, g \in \mathcal{D}$, define

$$\mathcal{D}(f, g) := \frac{1}{\pi}\int_{\mathbb{D}} f'(z)\overline{g'(z)}\,dA(z).$$

This is a semi-inner product, and clearly $\mathcal{D}(f, f) = \mathcal{D}(f)$. In particular, $\mathcal{D}(f)^{1/2}$ is a semi-norm on $\mathcal{D}$. It is not quite a norm, since $\mathcal{D}(f) = 0$ whenever f is a constant. To get round this, we define

$$\langle f, g\rangle_{\mathcal{D}} := \langle f, g\rangle_{H^2} + \mathcal{D}(f, g) \qquad (f, g \in \mathcal{D}),$$

where $\langle\cdot,\cdot\rangle_{H^2}$ is the usual inner product on H^2. This gives a genuine inner product on $\mathcal{D}$, and the corresponding norm $\|\cdot\|_{\mathcal{D}}$ is given by

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2}^2 + \mathcal{D}(f) \qquad (f \in \mathcal{D}).$$

Theorem 1.1.4 *The Dirichlet space $\mathcal{D}$ is a Hilbert space with respect to the norm $\|\cdot\|_{\mathcal{D}}$.*

Proof Writing $f(z) = \sum_{k\geq 0} a_k z^k$, we have $\|f\|_{\mathcal{D}}^2 = \sum_{k\geq 0}(k+1)|a_k|^2$. Thus the map $f \mapsto ((k+1)^{1/2}a_k)_{k\geq 0}$ is an isometry of $\mathcal{D}$ onto ℓ^2, the space of square summable sequences. As ℓ^2 is a Hilbert space, so too is $\mathcal{D}$. $\square$

This is not the only way to make $\mathcal{D}$ a Hilbert space. Another common choice is to take $\|f\|^2 := |f(0)|^2 + \mathcal{D}(f)$. This also gives a Hilbert-space norm on $\mathcal{D}$, equivalent to $\|\cdot\|_{\mathcal{D}}$. However, unless stated otherwise, we shall always assume that $\mathcal{D}$ carries the norm $\|\cdot\|_{\mathcal{D}}$.

Exercises 1.1

1. Let $f \in \mathcal{D}$. Prove that $\|f\|_{\mathcal{D}}^2 \le |f(0)|^2 + 2\mathcal{D}(f)$.
2. Let $f \in \mathcal{D}$, say $f(z) = \sum_{k\ge0} a_k z^k$.
   - (i) Let $s_n(z) := \sum_{k=0}^n a_k z^k$. Prove that $\|s_n\|_{\mathcal{D}} \le \|f\|_{\mathcal{D}}$ for all $n \ge 0$, and that $\|s_n - f\|_{\mathcal{D}} \to 0$ as $n \to \infty$.
   - (ii) Let $f_r(z) := f(rz)$. Prove that $\|f_r\|_{\mathcal{D}} \le \|f\|_{\mathcal{D}}$ for all $0 < r < 1$, and that $\|f_r - f\|_{\mathcal{D}} \to 0$ as $r \to 1^-$.
3. Let $(f_n)_{n\ge1} \in \mathcal{D}$. Show that, if $f_n \to f$ locally uniformly on $\mathbb{D}$, then we have $\mathcal{D}(f) \le \liminf_{n\to\infty} \mathcal{D}(f_n)$. Deduce that, if $\sup_n \mathcal{D}(f_n) < \infty$, then $f \in \mathcal{D}$.
4. The *analytic Wiener algebra* is $W^+ := \{\sum_{k\ge0} a_k z^k : \sum_{k\ge0} |a_k| < \infty\}$. Give examples to show that $\mathcal{D} \not\subset W^+$ and that $W^+ \not\subset \mathcal{D}$.
5. The *disk algebra* $A(\mathbb{D})$ is the set of continuous functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ that are holomorphic in $\mathbb{D}$. Give examples to show that $\mathcal{D} \not\subset A(\mathbb{D})$ and that $A(\mathbb{D}) \not\subset \mathcal{D}$.
6. (This exercise and the next make use of the Hardy spaces H^p for general p. For a summary of their properties, see Appendix A.) Let $f \in \mathrm{Hol}(\mathbb{D})$.
   - (i) Show that, if $f' \in H^2$, then $f \in \mathcal{D}$.
   - (ii) Show that, if $f' \in H^1$, then $f \in \mathcal{D}$. [This is somewhat harder. Use Hardy's inequality (Theorem A.1.9) to show that, if $f' \in H^1$, then the Taylor coefficients a_k of f satisfy $\sum_k |a_k| < \infty$ and $a_k = O(1/k)$.]
   - (iii) Let $f(z) := \sum_{k\ge1} z^k/k$. Show that $f' \in H^p$ for all $p < 1$ but $f \notin \mathcal{D}$.
7. We know that $\mathcal{D} \subset H^2$. Show that in fact $\mathcal{D} \subset \bigcap_{p<\infty} H^p$, as follows. Fix $f \in \mathcal{D}$ and $p \in (2, \infty)$.
   - (i) Let $q \in (1, 2)$ be chosen so that $1/p + 1/q = 1$. Use Hölder's inequality to show that the Taylor coefficients a_k of f satisfy
   $$\sum_k |a_k|^q \le \Big(\sum_k \frac{1}{k^{q/(2-q)}}\Big)^{(2-q)/2} \Big(\sum_k k|a_k|^2\Big)^{q/2} < \infty.$$
   - (ii) Use the Hausdorff–Young inequality to deduce that $f \in H^p$.

1.2 Reproducing kernels

We begin this section with a simple observation.

Theorem 1.2.1 *If $f \in \mathcal{D}$, then*

$$|f(z) - f(0)|^2 \leq \mathcal{D}(f) \log\Big(\frac{1}{1-|z|^2}\Big) \qquad (z \in \mathbb{D}).$$

Proof Write $f(z) = \sum_{k\geq 0} a_k z^k$. For each $z \in \mathbb{D}$, the Cauchy–Schwarz inequality gives

$$|f(z) - f(0)|^2 = \Big|\sum_{k\geq 1} a_k z^k\Big|^2 \leq \Big(\sum_{k\geq 1} k|a_k|^2\Big)\Big(\sum_{k\geq 1} \frac{|z|^{2k}}{k}\Big) = \mathcal{D}(f) \log\Big(\frac{1}{1-|z|^2}\Big). \quad \square$$

This result has some interesting consequences. The first is a growth estimate.

Corollary 1.2.2 *If $f \in \mathcal{D}$, then*

$$f(z) = o\Big(\Big(\log \frac{1}{1-|z|^2}\Big)^{1/2}\Big) \qquad (|z| \to 1^-).$$

Proof Let $f(z) := \sum_{k\geq 0} a_k z^k$ and $s_n(z) := \sum_{k=0}^n a_k z^k$. Clearly s_n is bounded on $\mathbb{D}$. Hence, applying Theorem 1.2.1 to $f - s_n$, we obtain

$$\limsup_{|z|\to 1^-} |f(z)|\Big(\log \frac{1}{1-|z|^2}\Big)^{-1/2} \leq \mathcal{D}(f - s_n)^{1/2}.$$

The result follows upon letting $n \to \infty$. $\square$

One might ask whether it is possible to go further, and prove that f must be bounded on $\mathbb{D}$. This is not the case. For example, consider the function

$$f(z) := \sum_{k\geq 2} \frac{z^k}{k \log k} \qquad (z \in \mathbb{D}).$$

Then

$$\mathcal{D}(f) = \sum_{k\geq 2} k\Big(\frac{1}{k \log k}\Big)^2 = \sum_{k\geq 2} \frac{1}{k(\log k)^2} < \infty,$$

so $f \in \mathcal{D}$ but, on the other hand,

$$\liminf_{r\to 1^-} f(r) \geq \sum_{k\geq 2} \frac{1}{k \log k} = \infty,$$

so f is unbounded on $\mathbb{D}$. (In fact, much more is true: see Exercise 1.2.2 below.)

A second consequence of Theorem 1.2.1 is that, for each $w \in \mathbb{D}$, the evaluation map $f \mapsto f(w)$ is a continuous linear functional on $\mathcal{D}$. By the Riesz representation theorem for Hilbert spaces, this functional is given by taking the inner product with some function $k_w \in \mathcal{D}$. We now give a direct proof of this, at the same time identifying k_w.

Theorem 1.2.3 *For $w \in \mathbb{D} \setminus \{0\}$, define*

$$k_w(z) := \frac{1}{z\overline{w}} \log\Big(\frac{1}{1 - z\overline{w}}\Big) \qquad (z \in \mathbb{D}),$$

and set $k_0 \equiv 1$. Then $k_w \in \mathcal{D}$ and

$$f(w) = \langle f, k_w \rangle_{\mathcal{D}} \qquad (w \in \mathbb{D}).$$

Proof Fix $w \in \mathbb{D}$. With k_w defined as in the statement of the theorem, we have $k_w(z) = \sum_{j\geq 0}(\overline{w}^j/(j+1))z^j$, the sum converging in the norm of $\mathcal{D}$. Therefore $k_w \in \mathcal{D}$. Further, if $f \in \mathcal{D}$, say $f(z) = \sum_{j\geq 0} a_j z^j$, then

$$\langle f, k_w \rangle_{\mathcal{D}} = \sum_{j\geq 0}(j+1)a_j \frac{w^j}{j+1} = \sum_{j\geq 0} a_j w^j = f(w). \qquad \square$$

The functions k_w are called *reproducing kernels*. They depend not only on the space $\mathcal{D}$ but also on our choice of inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$. Later, we shall see that they play an important role in the study of zero sets (see §4.2) and Pick interpolation (see §5.3).

Exercises 1.2

1. Show that, for each $w \in \mathbb{D} \setminus \{0\}$,

$$\|k_w\|^2_{\mathcal{D}} = \frac{1}{|w|^2} \log\Big(\frac{1}{1 - |w|^2}\Big).$$

2. Let $\epsilon : \mathbb{D} \to (0, \infty)$ be a function such that $\liminf_{|z|\to 1^-} \epsilon(z) = 0$. Show that there exists $f \in \mathcal{D}$ such that

$$f(z) \neq O\Big(\epsilon(z)\Big(\log \frac{1}{1 - |z|^2}\Big)^{1/2}\Big) \qquad (|z| \to 1).$$

[Hint: Let (w_n) be a sequence in $\mathbb{D}$ such that $|w_n| \to 1$ and $\epsilon(w_n) \to 0$. Use the result of Exercise 1 to show that the sequence

$$g_n := \epsilon(w_n)^{-1}\Big(\log \frac{1}{1 - |w_n|^2}\Big)^{-1/2} k_{w_n} \qquad (n \geq 1)$$

is unbounded in $\mathcal{D}$. By the Banach–Steinhaus theorem, there exists $f \in \mathcal{D}$ such that $\sup_n |\langle f, g_n \rangle_{\mathcal{D}}| = \infty$.]

1.3 Multiplication

Given $f, g \in \mathcal{D}$, we can form their pointwise product fg. It is natural to ask whether $fg \in \mathcal{D}$. The answer turns out to be negative.

Theorem 1.3.1 *The Dirichlet space is not an algebra.*

Proof We construct an explicit function f such that $f \in \mathcal{D}$ but $f^2 \notin \mathcal{D}$. Set

$$f(z) := \sum_{k\geq 2} \frac{z^k}{k(\log k)^{3/4}} \qquad (z \in \mathbb{D}).$$

By Theorem 1.1.2, we have

$$\mathcal{D}(f) = \sum_{k\geq 2} \frac{k}{(k(\log k)^{3/4})^2} = \sum_{k\geq 2} \frac{1}{k(\log k)^{3/2}} < \infty,$$

so $f \in \mathcal{D}$. On the other hand, $f(z)^2 = \sum_{k\geq 4} a_k z^k$ where

$$\begin{aligned} a_k &= \sum_{j=2}^{k-2} \frac{1}{j(\log j)^{3/4}(k-j)(\log(k-j))^{3/4}} \\ &\geq \frac{1}{k(\log k)^{3/4}} \sum_{j=2}^{k-2} \frac{1}{j(\log j)^{3/4}} \\ &\geq \frac{C}{k(\log k)^{1/2}}, \end{aligned}$$

for some constant $C > 0$ (see Exercise 1.3.1). Hence, by Theorem 1.1.2 again,

$$\mathcal{D}(f^2) \geq C^2 \sum_{k\geq 4} \frac{1}{k \log k} = \infty,$$

and so $f^2 \notin \mathcal{D}$. □

We do however have a positive result. In what follows, H^∞ denotes the algebra of bounded holomorphic functions on the unit disk with the norm $\|f\|_{H^\infty} := \sup_{z\in\mathbb{D}} |f(z)|$.

Theorem 1.3.2 *The space $\mathcal{D} \cap H^\infty$ is a Banach algebra with respect to the norm*

$$\|f\|_{\mathcal{D}\cap H^\infty} := \|f\|_{H^\infty} + \mathcal{D}(f)^{1/2}. \qquad (1.2)$$

Proof Let $f, g \in \mathcal{D} \cap H^\infty$. Using Minkowski's inequality, we have

$$\begin{aligned}\mathcal{D}(fg)^{1/2} &= \Big(\frac{1}{\pi}\int_{\mathbb{D}} |f'g + fg'|^2\,dA\Big)^{1/2}\\ &\le \Big(\frac{1}{\pi}\int_{\mathbb{D}} |f'g|^2\,dA\Big)^{1/2} + \Big(\frac{1}{\pi}\int_{\mathbb{D}} |fg'|^2\,dA\Big)^{1/2}\\ &\le \mathcal{D}(f)^{1/2}\|g\|_{H^\infty} + \|f\|_{H^\infty}\mathcal{D}(g)^{1/2}.\end{aligned}$$

From this, it follows that $\mathcal{D}\cap H^\infty$ is an algebra, and that (1.2) defines an algebra norm on it. Finally, as both $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $(H^\infty, \|\cdot\|_{H^\infty})$ are complete spaces, it is easy to check that $(\mathcal{D}\cap H^\infty, \|\cdot\|_{\mathcal{D}\cap H^\infty})$ is complete. □

In the light of this theorem, it is tempting to believe that $\mathcal{D}$ is stable under multiplication by elements of $\mathcal{D}\cap H^\infty$. We shall see later that this is actually false. The functions that do have this important property are called multipliers, and we shall return to study them in detail in Chapter 5.

Exercises 1.3

1. Let $\alpha \in (0, 1)$. By comparing the sum with an integral, show that

$$\sum_{j=2}^{k} \frac{1}{j(\log j)^\alpha} = \frac{(\log k)^{1-\alpha}}{1-\alpha} + O(1) \quad \text{as } k \to \infty.$$

1.4 Composition

This short section is based on the following conformal invariance property.

Theorem 1.4.1 *Let D_1, D_2 be domains, let $\phi : D_1 \to D_2$ be a conformal mapping and let $f : D_2 \to \mathbb{C}$ be a holomorphic function. Then*

$$\int_{D_1} |(f\circ\phi)'(z)|^2\,dA(z) = \int_{D_2} |f'(w)|^2\,dA(w).$$

Proof Making the substitution $w = \phi(z)$, we have $dA(w) = |\phi'(z)|^2\,dA(z)$, whence

$$\int_{D_2} |f'(w)|^2\,dA(w) = \int_{D_1} |f'(\phi(z))|^2\,|\phi'(z)|^2\,dA(z) = \int_{D_1} |(f\circ\phi)'(z)|^2\,dA(z). \quad □$$

In particular, taking $D_1 = \mathbb{D}$ and $f(z) = z$, we obtain the following interpretation of the Dirichlet integral.

Corollary 1.4.2 *Let $\phi : \mathbb{D} \to \mathbb{C}$ be an injective holomorphic map. Then $\mathcal{D}(\phi)$ equals $1/\pi$ times the area of $\phi(\mathbb{D})$.*

Perhaps the most important case of Theorem 1.4.1 is when $D_1 = D_2 = \mathbb{D}$, in other words, when ϕ is a conformal automorphism of $\mathbb{D}$. The automorphisms of $\mathbb{D}$ are precisely the Möbius transformations of the form

$$\phi(z) = e^{i\theta}\frac{a-z}{1-\overline{a}z} \qquad (a \in \mathbb{D},\ |e^{i\theta}| = 1).$$

We write $\mathrm{Aut}(\mathbb{D})$ for this family of functions.

Corollary 1.4.3 *If $f \in \mathrm{Hol}(\mathbb{D})$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, then $\mathcal{D}(f \circ \phi) = \mathcal{D}(f)$. Consequently, if $f \in \mathcal{D}$, then also $f \circ \phi \in \mathcal{D}$.*

The same argument shows that, if $f \in \mathcal{D}$ and $\phi : \mathbb{D} \to \mathbb{D}$ is any injective holomorphic function, then $f \circ \phi \in \mathcal{D}$. To what extent can the hypothesis ‘injective’ be weakened? We shall return to this question in Chapter 6. In the same chapter we shall also see that, quite remarkably, the property of Möbius invariance described in Corollary 1.4.3 essentially characterizes the Dirichlet space.

Exercises 1.4

1. Use Corollary 1.4.2 to give another example of an unbounded function in $\mathcal{D}$.

1.5 Douglas’ formula

Let $\mathbb{T}$ denote the unit circle. Given $f \in \mathrm{Hol}(\mathbb{D})$ and $\zeta \in \mathbb{T}$, we write $f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$, whenever this radial limit exists. If $f \in H^2$, then $f^*(\zeta)$ exists a.e. on $\mathbb{T}$. Moreover, $f^* \in L^2(\mathbb{T})$ and f is the Poisson integral of f^*. The correspondence $f \leftrightarrow f^*$ allows us to view the Hardy space as a space of functions on the unit circle, and this turns out to be vital for many applications.

There is a formula for $\mathcal{D}(f)$ expressed purely in terms of f^*. It is due to Douglas [39].

Theorem 1.5.1 (Douglas’ formula) *Let $f \in H^2$. Then*

$$\mathcal{D}(f) = \frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\frac{f^*(\lambda) - f^*(\zeta)}{\lambda - \zeta}\right|^2 |d\lambda|\,|d\zeta|.$$

We shall deduce this formula from a lemma about L^2-Fourier series. Given a function $\phi \in L^2(\mathbb{T})$, we write $\widehat{\phi}(k)$ for its k-th Fourier coefficient, namely

$$\widehat{\phi}(k) := \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it})e^{-ikt}\,dt \qquad (k \in \mathbb{Z}).$$

In this notation, Parseval's formula becomes

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{it})|^2\,dt = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(k)|^2.$$

Lemma 1.5.2 *Let $\phi \in L^2(\mathbb{T})$. Then*

$$\frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big|\frac{\phi(\lambda)-\phi(\zeta)}{\lambda-\zeta}\Big|^2\,|d\lambda|\,|d\zeta| = \sum_{k\in\mathbb{Z}} |k|\,|\widehat{\phi}(k)|^2.$$

Proof After the change of variables $\lambda = e^{i(s+t)}$, $\zeta = e^{it}$, the double integral becomes

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big|\frac{\phi(e^{i(s+t)})-\phi(e^{it})}{e^{is}-1}\Big|^2\,dt\,ds.$$

Parseval's formula, applied to the function $\zeta \mapsto \phi(e^{is}\zeta)-\phi(\zeta)$, gives

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{i(s+t)})-\phi(e^{it})|^2\,dt = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(k)|^2|e^{iks}-1|^2.$$

Hence the double integral equals

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{k\in\mathbb{Z}} |\widehat{\phi}(k)|^2\frac{|e^{iks}-1|^2}{|e^{is}-1|^2}\,ds.$$

Finally, we remark that, for each integer $k \neq 0$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{e^{iks}-1}{e^{is}-1}\Big|^2\,ds = \frac{1}{2\pi}\int_0^{2\pi}|1+e^{is}+\cdots+e^{i(|k|-1)s}|^2\,ds = |k|,$$

the last equality again from Parseval's formula. The result follows. □

Proof of Theorem 1.5.1 We apply Lemma 1.5.2 with $\phi = f^*$. Observe that, writing $f(z) = \sum_{k\geq 0} a_k z^k$, we have $\widehat{f^*}(k) = a_k$ if $k \geq 0$ and $\widehat{f^*}(k) = 0$ if $k < 0$. Hence

$$\frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big|\frac{f^*(\lambda)-f^*(\zeta)}{\lambda-\zeta}\Big|^2\,|d\lambda|\,|d\zeta| = \sum_{k\geq 0} k|a_k|^2.$$

By Theorem 1.1.2, this last expression equals $\mathcal{D}(f)$. □

As mentioned at the beginning of the section, if $f \in H^2$, then the radial limit f^* exists a.e. Indeed, for almost every $\zeta \in \mathbb{T}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each non-tangential approach region $|z-\zeta| < \kappa(1-|z|)$. For functions in $\mathcal{D}$, the same is true even for certain tangential approach regions. We shall prove this as an application of Douglas' formula.

Theorem 1.5.3 *Let $f \in \mathcal{D}$. Then, for a.e. $\zeta \in \mathbb{T}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each oricyclic approach region $|z-\zeta| < \kappa(1-|z|)^{1/2}$.*

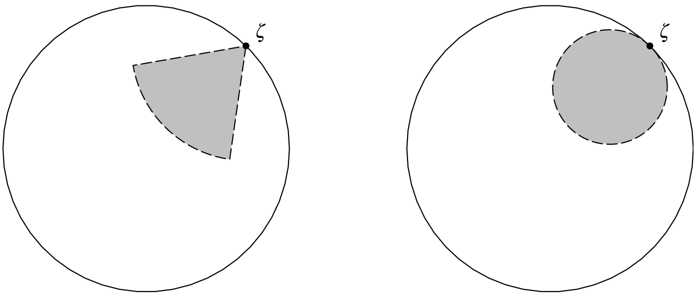

Figure 1.1 Non-tangential and oricyclic approach regions

Lemma 1.5.4 *If* $g \in H^2$, *then* $\lim_{|z|\to 1}(1-|z|^2)|g(z)|^2 = 0$.

Proof Write $g(z) = \sum_{k\ge 0} a_k z^k$. By the Cauchy–Schwarz inequality,

$$|g(z)|^2 \le \sum_{k\ge 0} |a_k|^2 \sum_{l\ge 0} |z|^{2l} = \|g\|_{H^2}^2 (1-|z|^2)^{-1} \qquad (z \in \mathbb{D}).$$

Hence $\limsup_{|z|\to 1}(1-|z|^2)|g(z)|^2 \le \|g\|_{H^2}^2$. Replacing g by $g - \sum_{k=0}^n a_k z^k$, and then letting $n \to \infty$, we deduce that in fact $\lim_{|z|\to 1}(1-|z|^2)|g(z)|^2 = 0$. □

Proof of Theorem 1.5.3 Let $f \in \mathcal{D}$. By Douglas' formula, we have

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f^*(\lambda) - f^*(\zeta)}{\lambda - \zeta}\right|^2 |d\lambda|\,|d\zeta| < \infty.$$

Hence, for almost every $\zeta \in \mathbb{T}$, the radial limit $f^*(\zeta)$ exists and satisfies

$$\int_{\mathbb{T}} \left|\frac{f^*(\lambda) - f^*(\zeta)}{\lambda - \zeta}\right|^2 |d\lambda| < \infty.$$

Fix such a ζ, and define

$$g(z) := \frac{f(z) - f^*(\zeta)}{z - \zeta} \qquad (z \in \mathbb{D}).$$

Then $g \in \mathrm{Hol}(\mathbb{D})$, the radial limit g^* exists a.e., and $g^* \in L^2(\mathbb{T})$. Since the denominator of g is an outer function, Smirnov's maximum principle implies that $g \in H^2$ (see Theorem A.3.8 in Appendix A). Therefore Lemma 1.5.4 applies, and $|g(z)|^2 = o((1-|z|^2)^{-1})$ as $|z| \to 1$. It follows that

$$|f(z) - f^*(\zeta)|^2 = |z-\zeta|^2 |g(z)|^2 = o\Big(\frac{|z-\zeta|^2}{1-|z|^2}\Big) \qquad (|z| \to 1).$$

Hence $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each approach region $|z-\zeta| < \kappa(1-|z|)^{1/2}$. □