THE COSMOLOGICAL SINGULARITY

Written for researchers focusing on general relativity, supergravity, and cosmology, this is a self-contained exposition of the structure of the cosmological singularity in generic solutions of the Einstein equations, and an up-to-date mathematical derivation of the theory underlying the Belinski–Khalatnikov–Lifshitz (BKL) conjecture on this field.

Part I provides a comprehensive review of the theory underlying the BKL conjecture. The generic asymptotic behavior near the cosmological singularity of the gravitational field, and fields describing other kinds of matter, is explained in detail. Part II focuses on the billiard reformulation of the BKL behavior. Taking a general approach, this section does not assume any simplifying symmetry conditions and applies to theories involving a range of matter fields and space-time dimensions, including supergravities.

Overall, this book will equip theoretical and mathematical physicists with the theoretical fundamentals of the Big Bang, Big Crunch, Black Hole singularities, their billiard description, and emergent mathematical structures.

Vladimir Belinski holds a permanent professor position at the International Center for Relativistic Astrophysics Network, Italy. He is noted for his role in several key developments in theoretical physics, including the Belinski–Khalatnikov–Lifshitz conjecture on the behavior of generic solutions of Einstein equations near a cosmological singularity, the Belinski–Zakharov transform, and the "Inflationary Attractor." He is co-author of the book *Gravitational Solitons* (Cambridge University Press, 2001) and has received the Landau Prize of the Russian Academy of Sciences (1974) and Marcel Grossmann Award (2012).

Marc Henneaux is Full Professor at the Université Libre de Bruxelles and Director of the International Solvay Institutes for Physics and Chemistry. He has contributed significantly to understandings of gravity and black holes in three dimensions, and the geometrical and algebraic aspects of theories with a gauge freedom (in particular the Becchi–Rouet– Stora–Tyutin symmetry and its cohomology). He has received the Francqui Prize (2000), the Humboldt Prize (2009), the Bogoliubov Prize (2014), the FNRS Quinquennial Prize in Fundamental Exact Sciences (2015), and two ERC advanced grants (2011–2015 and 2016–2021).

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Preface

The first exactly solvable cosmological models of Einstein's theory revealed the presence of a very striking phenomenon: the Big Bang singularity. Since the time it was discovered in 1922 by Alexander Friedmann, a fundamental question has arisen as to whether this phenomenon is due to the special simplifying assumptions underlying the exactly solvable models or whether a singularity is a general property of the Einstein equations. This question was formulated for the first time by L. Landau in 1959.

The question was answered by V. Belinski, I. Khalatnikov and E. Lifshitz (BKL) in 1969. The BKL work showed that a singularity is a general property of a generic cosmological solution of the classical gravitational equations and not a consequence of the special symmetric structure of the exact models. Most importantly, BKL were able to find the analytical structure of this generic solution and show that its behavior is of an extremely complex oscillatory character, of chaotic type. Because it provides the description of a general solution of the Einstein equations (i.e., a solution depending on sufficiently many freely adjustable functions of space), the BKL analysis sheds light on intrinsic properties of Einstein gravity. Given the nonlinear character of the Einstein equations and the difficulty of finding exact solutions without symmetries, the BKL results are quite notable. They have a fundamental significance not only for cosmology but also for the evolution of collapsing matter forming a black hole. The last stage of collapsing matter will follow in general the BKL regime.

The chaotic oscillations discovered by BKL can be understood in terms of a "cosmological billiard" system, where the cosmological evolution is described at each spatial point as the relativistic motion of a fictitious billiard ball in the Lorentzian space of the logarithmic scale factors. This reformulation of the BKL behavior can be naturally extended to arbitrary matter couplings and dimensions of space-time, enabling one to show that the BKL regime is inherent not only to General Relativity but also to more general physical theories containing gravity, such as supergravity models. The dimension of the billiard table and the nature of the walls that bound it depend on the theory, but the billiard description remains universally valid.

The billiard point of view provides a remarkably simple description of the gravitational field in the vicinity of a spacelike singularity. In spite of the complexity of the Einstein-matter field equations, the asymptotic behavior of the fields near a cosmological singularity can be phrased in surprisingly elementary terms involving finite-dimensional dynamical systems. This description is valid generically, i.e., without making any symmetry assumption.

The billiard point of view has also unexpectedly led to the discovery of a remarkable connection with one of the most beautiful and active subjects of modern mathematics, namely hyperbolic Coxeter groups and the theory of indefinite Kac–Moody algebras. This connection emerges because the billiard region in which the cosmological billiard ball moves turns out to possess exceptional properties, which imply that the group of reflections in the billiard walls is a simplex crystallographic hyperbolic Coxeter group for the known theories containing gravity. This intriguing fact opens up the fascinating perspective that an underlying infinite-dimensional symmetry algebra might play a central role in the fundamental formulation of gravity. However, at the time of writing this book, a complete proof of the presence of such algebras has not been found, so that the origin of the observed emergence of the hyperbolic Coxeter groups in the BKL description remains something of a mystery.

The purpose of this book is to explain at length the BKL analysis, starting from the early work on the subject and going all the way to the most modern developments.

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