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978-1-107-04721-1 - The Riemann Hypothesis for Function Fields: Frobenius Flow  
and Shift Operators

Machiel Van Frankenhuysen

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