THE BETHE WAVEFUNCTION

Michel Gaudin's book *La fonction d'onde de Bethe* is a uniquely influential masterpiece on exactly solvable models of quantum mechanics and statistical physics. Available in English for the first time, this translation brings his classic work to a new generation of graduate students and researchers in physics. It presents a mixture of mathematics interspersed with powerful physical intuition, retaining the author's unmistakably honest tone.

The book begins with the Heisenberg spin chain, starting from the coordinate Bethe Ansatz and culminating in a discussion of its thermodynamic properties. Delta-interacting bosons (the Lieb–Liniger model) are then explored, and extended to exactly solvable models associated with a reflection group. After discussing the continuum limit of spin chains, the book covers six- and eight-vertex models in extensive detail, from their lattice definition to their thermodynamics. Later chapters examine advanced topics such as multicomponent delta-interacting systems, Gaudin magnets and the Toda chain.

MICHEL GAUDIN is recognized as one of the foremost experts in this field, and has worked at Commissariat à l'Énergie Atomique (CEA) and the Service de Physique Théorique, Saclay. His numerous scientific contributions to the theory of exactly solvable models are well known, including his famous formula for the norm of Bethe wavefunctions.

JEAN-SÉBASTIEN CAUX is a Professor in the theory of low-dimensional quantum condensed matter at the University of Amsterdam. He has made significant contributions to the calculation of experimentally observable dynamical properties of these systems.

Tu ergo, Domine, fecisti ea, qui pulcher es: pulchra sunt enim; qui bonus es: bona sunt enim; qui es: sunt enim. Nec ita pulchra sunt, nec ita bona sunt, nec ita sunt, sicut Tu conditor eorum; quo comparata, nec pulchra sunt, nec bona sunt, nec sunt. Scimus haec, gratias Tibi, et scientia nostra scientiae Tuae comparata ignorantia est.

Saint AUGUSTINE

THE BETHE WAVEFUNCTION

MICHEL GAUDIN

Translated from the French original
'La fonction d'onde de Bethe' (1983)
by Jean-Sébastien Caux

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Foreword

It is my pleasure to welcome this translation of the original French version of my book *La fonction d'onde de Bethe* into English.

The theory of exactly solvable models, perhaps more than any other subfield of many-body physics, has the distinct advantage of providing solid, reliable and long-lasting knowledge. This latter characteristic perhaps explains why my original text, which is by now over three decades old, is still used by members of the scientific community, despite too many mistakes and neglects on my part. This is why the present work of J.-S. Caux is more than a translation, for by his revision he has drawn 'new from old' thanks to his style and rigour.

Despite all the developments which the field has known in the intervening period, and which are of course not treated or mentioned here, the fact probably remains that much of what is presented has not been too deprecated in the years since the original version appeared. I am convinced that this translation will bring to a much larger readership an accurate image of the status of the knowledge on these fascinating models at the moment of publication of the original. The initiative and merit belong to the translator, to whom I express my gratitude.

Michel Gaudin
Paris, April 2013

Translator's note

Michel Gaudin's 1983 book *La fonction d'onde de Bethe* remains, to the people who have read it, a uniquely influential masterpiece on the subject of exactly solvable models of quantum mechanics. As a beginning Ph.D. student in the mid-1990s, I remember coming across this rather intriguing work, which stood out by being already out of print, and in French. My experience with this book, which sparked my own personal interest in integrable models, is similar to that of many members of the community. Not only the contents but also the *style* of presentation are truly unique: a mixture of mathematical economy interspersed with powerful physical intuition, written in the author's unmistakably humble and honest tone.

Most of the motivation for performing this translation came from a desire to share the high-quality, timeless knowledge contained in the original book with a new generation of researchers. Besides, it is likely that the translation of this work into Russian in the late 1980s significantly contributed to establishing that country's strong and ongoing tradition in the field of exactly solvable models; the sad fact that the book remained inaccessible for other readers seemed too great an omission not to correct. Despite the fact that many important developments have taken place in the field of exactly solvable models since the original publication, the foundations of the theory of integrable models remain essentially unchanged, and the work of Michel Gaudin still stands as a unique 'snapshot' of the field as it was at an important moment in its history.

Finally, one of my hopes with this endeavour was to help make the many unknown (often rediscovered, but too often uncredited) contributions of Michel Gaudin more widely and justly recognized by the wider community.

I am grateful to my team in Amsterdam, namely Rianne van den Berg, Giuseppe Brandino, Michael Brockmann, Jacopo De Nardis, Sebas Eliëns, Davide Fioretto, Thessa Fokkema, Rob Hagemans, Jorn Mossel, Miłosz Panfil, Olya Shevchuk, Rogier Vlijm and Bram Wouters for pointing out many typographical errors as this translation was being prepared. I also thank Stijn de Baerdemacker, Patrick Dorey, Fabian Essler, Holger Frahm, Dimitri Gangardt, Paul Johnson, Austen Lamacraft, Barry McCoy, Giuseppe Mussardo and Maxim Olshanii for their encouragement and feedback.

I am particularly indebted to Vincent Pasquier, who helped this project from the very beginning and provided much useful feedback throughout.

I also thank Professor James McGuire for his feedback on the text and for sharing his reminiscences of his friendship and collaboration with Michel Gaudin. These helped put the project in its proper historical and personal perspective.

Last, but without doubt most importantly, I am extremely grateful to Michel Gaudin himself for his support of the project, his suggestions for improvements and for the extreme kindness with which he always welcomed my queries. It has been a privilege to be in direct contact with such an inspiring individual, both as a researcher and a human being.

Introduction

From the exact solution of the Ising model by Onsager in 1944 up to that of the hard hexagon model by Baxter in 1980, the statistical mechanics of two-dimensional systems has been enriched by a number of exact results. One speaks (in quick manner) of exact models once a convenient mathematical expression has been obtained for a physical quantity such as the free energy, an order parameter or some correlation, or at the very least once their evaluation is reduced to a problem of classical analysis. Such solutions, often considered as singular curiosities upon their appearance, often have the interest of illustrating the principles and general theorems rigorously established in the framework of definitive theories, and also enabling the control of approximate or perturbative methods applicable to more realistic and complex models. In the theory of phase transitions, the Ising model and the results of Onsager and Yang have eminently played such a reference role. With the various vertex models, the methods of Lieb and Baxter have extended this role and the collection of critical exponents, providing new useful elements of comparison with extrapolation methods, and forcing a refinement of the notion of universality. Intimately linked to two-dimensional classical models (but of less interest for critical phenomena), one-dimensional quantum models such as the linear magnetic chain and Bethe's famous solution have certainly contributed to the understanding of fundamental excitations in many-body systems. One could also mention the physics of one-dimensional conductors. All these theoretical physics questions, while undoubtedly representing the main motivation for research in solvable models, are not the main object of the present study, since their exposition would require a deep and diversified knowledge of physical theories. Here, the main focus is on methods particular to a family of models, in other words the techniques for their solution.

Despite an apparent diversity of definitions and methods, the models which have up to now been solved have intimate relationships. They can often be seen as special or limiting cases of a more general model encompassing them. This is the case for example of the ice model, various ferroelectrics, the discrete lattice gas or Ising model, all joined in the quadratic lattice eight-vertex model defined by Fan and Wu. Aside systems of dimers on planar lattices (periodic or not) treated by Pfaffian methods, exactly solvable models of two-dimensional statistical mechanics are brought back to one-dimensional quantum problems via the transfer matrix method; the static configurations of these nearest-neighbour interacting two-dimensional systems present a topological analogy with the spatio-temporal trajectories of a dynamical system in one space dimension. Whether one seeks to obtain the free energy of the classical system or the excitations of the quantum one, one is led to the calculation of a largest eigenvalue, a second-largest or an entire spectrum, in other words one must diagonalize a matrix, be it a Hamiltonian or a transfer matrix. The prototype of such a Hamiltonian is that of the chain of spin-1/2 atoms with nearest-neighbour exchange interactions, for which in 1931 Bethe discovered the eigenfunctions and spectrum by induction. These eigenfunctions or amplitudes are of a fundamentally original structure and take the form of sums over the various operations of a finite permutation group or, more generally, a reflection group. One could baptize them 'Bethe sums'; one rather speaks of Bethe's Ansatz or hypothesis. These amplitudes describe a non-diffractive 'scattering', in other words a totally elastic collision of the particles or spin waves concerned. This is accompanied by the conservation of as many 'quasi-momenta' as there are particles; the existence of these constants of motion leads to the complete integrability of the Hamiltonian system.

Bethe's method provides the unifying thread between the chapters of this book. Chapter 1 exposes the technique he used in his famous article from 1931, aiming to obtain the wavefunctions and energy spectrum of the Heisenberg–Ising anisotropic magnetic chain. The results of Bethe and Griffiths on the asymptotic localization of the roots of the coupled equations for the spectrum permit the classification of states and, in Chapter 2, the treatment of the chain's thermodynamics at arbitrary temperature. Yang's approach to the thermodynamics of bosons in one dimension was followed, giving a probably correct evaluation of the entropy which would still deserve a rigorous proof. The study of limiting cases contained in Chapter 3, including high- and low-temperature limits as well as the Ising limit, provides convincing evidence for the exactness of the results. Chapters 4 and 5 are dedicated to the δ-interacting Bose gas, which can be seen as a continuous limit of the Heisenberg–Ising spin chain. An extension of Bethe's technique is described therein, which views the N-body problem as that of the diffusion of a multidimensional wave by a kaleidoscope or ensemble of infinitely thin concurrent refracting plates invariant under the reflection group they generate. This is applied to the solution of the bosonic problem defined on a finite segment and to that of the open magnetic chain. Thereafter, a conjecture on the norm of Bethe wavefunctions is presented, constituting a prelude to the open question of correlation functions. Chapter 6 is an incursion into $(1 + 1)$-dimensional quantum field theory aiming to examine the connection between the integrable models of Luttinger and Thirring with a certain continuum limit of the anisotropic magnetic chain.

Vertex models on quadratic lattices are treated in Chapter 7 using Lieb's method for the diagonalization of the transfer matrix of the general neutrality condition-preserving six-vertex model. Only a sketchy presentation is given of the thermodynamics of the various models of ferroelectrics, and one must consult the extended review of Lieb and Wu to go any further. The solution of the (self-conjugate) eight-vertex model is described in Chapters 8 and 9, following for the most part Baxter's method. There again, the integrability of the transfer matrix or of the derived Hamiltonian with three anisotropy constants is linked to the existence of ternary relations between transfer micromatrices. These ternary relations, which can also be understood as star–triangle relations, constitute remarkable representations of permutations and lead to the existence of one-parameter commuting families, and thus to integrability.

The general theory of n-component δ-interacting particles occupies three chapters. The Bethe hypothesis for the real-space wavefunction is proven in Chapter 10; Yang's method intimately relates the realization of symmetry conditions to that of periodicity conditions. The analogy with the inverse scattering method is already manifest and Zamolodchikov's S-matrix algebra is similar to that of Yang's operators. Chapter 11 gives three pathways towards the solution of the fermionic system with two internal states. The first reduces the problem to an inhomogeneous six-vertex model. The second summarizes the algebraic method used in the author's thesis; the third is Faddeev's method giving an operatorial form of Bethe's sum, this being without doubt the most compact representation and that most susceptible to future developments. Chapter 12 exposes Sutherland's general solution for arbitrary wavefunction symmetry, together with various corollaries concerning in particular a remarkable basis for representation of permutations. Chapter 13 contains mixed results concerning certain completely integrable spin systems. Baxter's most recent solution to the ternary relations, which led him to that of the hard hexagon model, is described there. Chapter 14 is devoted to the study of the Toda chain as an integrable classical and quantum system, though it is not associated with Bethe's wavefunction in its primitive form.

Being concerned with a field subject to rapid and varied developments, the ensemble of these 14 chapters does not pretend to be either a synthesis or a review of results on models pertaining in one way or another to Bethe's method. The latter in any case now tends to be viewed in the framework of integrable models as a technique derived from the inverse scattering method. My point of view here is more concrete than general and remains that of a first internal publication in 1972 on Bethe's method for exactly solvable models, of which the present book is only a relative extension and update. It is limited to the treatment of finite or extensive systems of statistical mechanics, ignoring beautiful aspects of quantum field theory with the exception of the few elementary considerations of Chapter 6. I have simply ordered the questions arousing my interest for the benefit of those who might feel attracted by the ingeniosity of the construction and the unifying character of Bethe's method. The mathematical tools are simple, and on the subject of rigour, one can justly be more demanding; one must then refer to the original publications and to the numerous recent works on these questions.

The stimulating atmosphere of the Service de Physique Théorique of the Centre d'Études Nucléaires de Saclay has allowed me to bring this manuscript to fruition, which I publish thanks to the supporting encouragements of my colleagues C. Itzykson, G. Ripka, M. L. Mehta and R. Balian. My interest in the subject was initiated by J. des Cloizeaux, to whom I owe more than I can say, and was renewed by Professor C. N. Yang whose intuition did not refrain from exerting itself in this one-dimensional field. My thanks go to all those who have taught me something on the subject, in particular to E. Brézin, B. McCoy, J. B. McGuire, B. Derrida, M. Takahashi, D. Chudnovsky and G. Chudnovsky; they go to Dr C. K. Lai with whom I have had the pleasure to collaborate at the State University of New York at Stony Brook, and to J. M. Maillard who has agreed to proofread a number of chapters.

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