Introduction

> Those who have meditated on the beauty and utility of the general method of Lagrange – who have felt the power and dignity of that central dynamical theorem which he deduced from a combination of the principle of virtual velocities with the principle of D'Alembert – and who have appreciated the simplicity and harmony which he introduced by the idea of the variation of parameters, must feel the unfolding of a central idea.
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> *W. R. Hamilton [41]*

Structures are to be found in various shapes and sizes for various purposes and uses. These range from the human-made structures of bridges carrying traffic, buildings housing offices, and airplanes carrying passengers all the way down to the biologic structures of cells and proteins carrying genetic information. Structural mechanics is concerned with the behavior of structures under the action of applied loads – their deformations and internal loads. We present, in the following chapters, versatile methods to tackle some of the most common (and most difficult) problems facing engineers in the analysis of structures. This volume specifically considers the situations where the loads vary in time such that inertia effects are important in computing the responses.

The modeling of the dynamic response of structures introduces many additional considerations not anticipated from a static analysis. It is therefore worth our while to say just a little about *structural dynamics* and its place in *structural analyses*. The subject of rigid-body dynamics treats physical objects as bodies that undergo motion without any change of shape. This has many applications: the movement of machine parts, the flight of an aircraft or space vehicle, the motion of the Earth and the planets. In many instances, however, the primary concern is dynamic response involving changes of shape. This is particularly so in the design of structures as encountered in automobiles, ships, aircraft, space vehicles, offshore platforms, buildings, and bridges. Dynamic response involving deformations is usually oscillatory in nature; the structure vibrates about a configuration of stable equilibrium. For example, suppose that a building structure is in a state of static equilibrium under the gravity loads acting on it; when subjected to wind loading, the structure oscillates about this position of static equilibrium. An airplane provides an example of oscillatory motion about an equilibrium configuration that involves rigid-body motion. When in flight, the whole system moves as a rigid body but is also subjected to oscillatory motion due to engine and aerodynamic loads.

With the increasing use being made of lightweight, high-strength materials, structures today are more susceptible than ever before to critical vibrations. Modern buildings and bridges are lighter and more flexible and are made of materials that provide much lower energy dissipation; all these contribute to more intense vibration responses. Dynamic analysis of structures is therefore important for modern structures and is likely to become even more so.

Structural Analysis and Models

The term *model* is widely used in many different contexts, but here we mean a representation of a physical system that may be used to predict the behavior of a system in some desired respect. The actual physical system for which the predictions are to be made is called the *prototype*.

There are two broad classes of models: physical models and mathematical models. The physical model resembles the prototype in appearance but is usually of a different size, may involve different materials, and frequently operates under loads, temperatures, and so on that differ from those of the prototype. The use of these models belongs to the category of "experimental methods of structural analysis." The mathematical model consists of one or more equations (and, more likely nowadays, a numerical finite-element model) that describe the behavior of the system of interest. The equations of the model are based on certain basic laws and principles of mechanics and usually involve simplifying assumptions. These models broadly belong to the category of "analytical methods of structural analysis" (which sounds a bit tautological). The equations themselves may or may not be solved on a computer.

With the development of a valid model, it is possible to predict the immediate and future behavior of the prototype under a set of specified inputs and to examine a priori the effect of various possible design modifications.

We deal with models of different types in this book. Our primary model for "solving the problem" is finite-element (FE)–based and represents the structure in terms of a finite number of discrete unknowns. This approach is chosen because it is easily implemented on a computer and is scalable to large systems. It is also true that current commercial FE codes are such that once the geometry, material properties, and so on are correctly specified, then very high quality models are produced that give high quality predictive capability, and this must inform how structural dynamics should now be done.

Two practical finite elements are emphasized: the frame FE and the solid FE. These represent extremes in a way: the frame FE embeds many structural assumptions about behavior of slender members and consequently is very efficient where applicable, and the solid FE has no structural assumptions and hence is applicable to frames, shells, and solids alike, but it can be computationally expensive to use. With these two element models available, almost any structural problem can be solved in the sense that given the geometry, material properties, loads, and so on, responses can be generated. This is where a different level of model enters, one that helps to explain the computed numbers; these are of the "simple model" type. That is, when trying to understand a complex system, it is quite useful (and arguably necessary) to have available these simple models – not as solutions per se but as organizational principles for seeing through the voluminous numbers produced by the FE codes. They identify the model parameters that play a significant role. Sometimes they are constructed to go deeper into the mechanics of a problem; for example, there are the plate and shell simple models that follow the structural consequence of the thickness being thin, but there are also the modal, spectral, and wave-propagation *analysis models* that give insight into how to view and understand structural dynamics.

Goals and Outline of the Book

The primary goals of this book are

- To develop solution methods, general enough and scalable enough, to solve "big dynamics problems"
- To develop methods of analysis to "make sense" of the generated solutions
- Because geometric nonlinearities are an intimate aspect of flexible structures, to make the solution and analysis methods general enough to seamlessly handle nonlinear problems

The book is divided into two parts roughly corresponding to the first two objectives. Part I develops the mechanics and computer models to handle general problems, Part II develops the analytical models. The nonlinear analyses are distributed throughout the chapters.

Chapter 1 introduces some foundational ideas in the dynamics of elastic systems, ideas such as resonance and damping. Chapter 2 develops the mechanics needed to handle large, complex systems; the key concept introduced is that of virtual work. The formulation is in terms of discretized systems and is general enough to be applicable to static/dynamic, linear/nonlinear, and conservative/nonconservative systems alike. The only restriction is that the system be discretized. Chapter 3 uses the Ritz method to convert continuous systems into discrete form and then formalizes the process via the FE method. Chapter 4 is an attempt to classify the various types of dynamic problems based on the space-time variation of their loadings; this sets the contexts for the computational tools required and for the types of analysis procedures introduced. Chapter 5 ends Part I with a review of some of the computer methods used to implement our models. The essential algorithms discussed in detail are those for time integration of simultaneous equations and for solving eigenvalue problems.

Part II is the compilation of analytical models: modal analysis and eigenvalue problems in Chapter 6, spectral analysis and strong formulations in Chapter 7, flexible plates and shells in Chapter 8, and wave propagation and high-frequency analysis in Chapter 9. Chapter 10 ends Part II with an introduction to the concept of the stability of the motion. This is fundamentally a nonlinear notion, and the preceding chapters are developed general enough to anticipate this.

There are a good number of example problems distributed throughout the chapters. A few are straightforward "finger exercises" in that they take a previously established result and apply it to some problem. A few others do direct extensions of some developed model or result. A very exciting new type of example problem is made available because of Chapters 3 and 5 and is the predominant type of example problem in Part II; here the computer programs are used to produce results, but the analysis challenge is to "explain" the results. This is akin to the experimental challenge of collecting data on some partially unfamiliar or unknown dynamic problem and then trying to explain the data. The objective of the example problem is not so much to explain this or that result per se, but to show how the programs can be interacted with to produce additional data to aid discovering the explanation of the results. Remember that unlike the experimental analogue, the FE solution can provide almost unlimited information about the solution presented in almost unlimited different forms. Therefore, having control of the postprocessing capabilities is an important aspect of the analysis.

In terms of philosophy, any FE program can be used for the underlying computations, but to affirm the integral aspect of the computations and the analyses, source code and executables are provided on the accompanying website: www.cambridge.org/doyle_structures_FEM. These codes and executables can be used to re-create the data used in the majority of the examples as well as extend them. The two major codes are `SDframe/SDsolid`; these are leaner versions of the programs used in the `QED` package [28]. Also, as additional encouragement to reproduce the results, all example problems are documented in terms of dimensions, material properties, boundary conditions, and loadings, as well as mesh information.

As a final point, structural analysis computer programs generally do not use a built-in system of units and do not utilize any dimensional conversion constants. Therefore, any consistent system of units may be used for input, and the corresponding calculated results are output in the same units. All the relevant data for the example problems are presented in both SI units and common units. Because both systems of units are used in this book, we generally prefer (when convenient) to present the results in nondimensional form; however, for units that are common to both systems (e.g., time, frequency, angle), they are left as is.

PART I

MECHANICS AND MODELS

1 Dynamics of Simple Elastic Systems

This chapter is concerned with the formulation of the equations of motion (EoM) of simple systems. What is meant by *simple* is that the systems have just a single degree of freedom (SDoF) and does not imply that the underlying mechanics is simple or elementary in any way.

The concept of vibration is fundamental to understanding the dynamics of elastic structures. The study of vibration is concerned with the oscillatory motion of bodies; all bodies with elasticity and mass are capable of exhibiting vibrations. Resonant (or natural) frequencies are the frequencies at which a structure exhibits relatively large response amplitudes for relatively small inputs. Even if the excitation forces are not sinusoidal, these frequencies tend to dominate the response. In practice, large resonant responses are mitigated by the presence of damping and nonlinear effects. Damping is considered in this chapter, whereas the effects of nonlinearities are distributed throughout the other chapters. The use of Fourier analysis (or spectral analysis) as a means of describing time-varying behavior is essential to the study of structural dynamics, and this too is developed in this chapter.

1.1 Motion of Simple Systems

This section reviews the dynamics of elastic systems in the form of a spring-mass-dashpot. We restrict the emphasis to concepts that are used directly in this and later chapters. References 45, 81, and 83 are good sources for additional details on the material covered here.

Newton's Laws for Moving Masses

Everyday experience of mass is as a weight, so much so that the words *mass* and *weight* are often used interchangeably. In dynamics, these two words are associated with quite distinct concepts – the first with inertia and the second with gravitation attraction. We elaborate on the difference between the two.

Consider a particle as an object with negligible dimensions but definite mass M and definite position in three-dimensional (3D) space $\hat{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$,

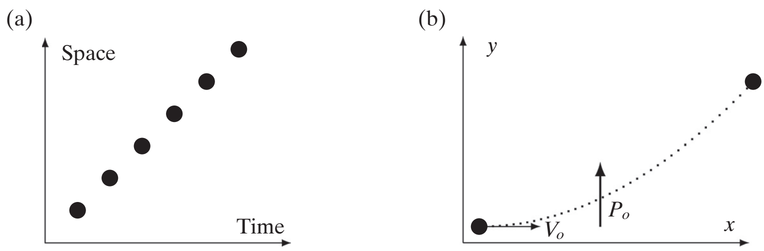

Figure 1.1. Motions of a particle: (a) uniform motion, no forces acting; (b) change of direction due to load.

and $\hat{k}$ are unit vectors in the x-, y-, and z-directions, respectively. Its position may change over time giving

$$\text{Velocity:} \qquad \frac{d\hat{r}(t)}{dt} = \dot{\hat{r}} = \hat{v}(t)$$

$$\text{Acceleration:} \qquad \frac{d^2\hat{r}(t)}{dt^2} = \ddot{\hat{r}} = \frac{d\hat{v}(t)}{dt} = \dot{\hat{v}} = \hat{a}(t)$$

This contrasts with real objects (a ball or a brick, say), which do have dimension (diameter, length, etc.) and therefore can also rotate or spin.

Newton's first law says that a particle continues to move in a straight line with uniform motion unless acted on by unbalanced forces; this is shown in Figure 1.1(a). Newton's second law says that the acceleration of a particle is proportional to the resultant force $\sum \hat{P}$ acting on it. This is expressed variously as

$$\sum \hat{P}(t) = M\hat{a}(t) = \frac{d}{dt}M\hat{v} = \frac{d\hat{p}}{dt}$$

where the proportionality constant is the mass, and $\hat{p} \equiv M\hat{v}$ is the *momentum*. This is a vector relation, so a particle with initial velocity V_o in the x-direction and acted on by a constant force P_o in the y-direction gives a trajectory

$$\hat{r}(t) = V_o t\hat{\imath} + \frac{1}{2}P_o t^2 \hat{\jmath} \quad \text{or} \quad y = \frac{P_o x^2}{2V_o}$$

The path followed is a curved trajectory in space, as shown in Figure 1.1(b). Mass is thus associated with inertia in the sense that if it is not moving, then effort must be exerted to get it moving; conversely, if it is moving, effort must be exerted to get it to stop or change direction.

Newton's third law says that the forces of action and reaction between interacting particles are equal in magnitude, opposite in direction, and collinear. The prime example is the force between two bodies in pressing contact; the gravitation force between two masses M_1 and M_2 is also a good example given by

$$P = \frac{GM_1M_2}{r^2} \qquad r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

where r is the separation distance, and $G = 66.73 \times 10^{-12}\,\mathrm{m}^3/(\mathrm{kg\,s}^2)$ is the universal gravitation constant. The surprising aspect of this is that a small object (a human, say) has the same gravitational effect on a large object (the moon, say) as the large object has on the small object. Another simple example of forces between objects is two masses connected by a spring as shown in Figure 1.2(a). If the spring is linear of natural length L_o and the separation between the masses is L, then the forces on the left and right masses are, respectively,

$$F_1 = +K(L - L_o) = +K(x_2 - x_1 - L_o) = +K(u_2 - u_1)$$
$$F_2 = -K(L - L_o) = -K(x_2 - x_1 - L_o) = -K(u_2 - u_1)$$

where K is the spring constant, and x_1, x_2, u_1, and u_2 are the particle positions and displacements, respectively.

All objects experience some sort of dissipation of energy (or damping) when set in motion. This is due to such factors as friction with the surrounding air or water and internal friction of the material itself. There are many models of damping; in the *viscous damping model*, for example, the retarding force is proportional to the instantaneous velocity, so $\hat{F}^d = C\dot{\hat{r}} = C\hat{v}$, where C is the damping constant.

In these examples of interaction forces, because the force depends on particle position and/or velocity, its effect on the motion is quite different from the effect if it were an external applied load. We illustrate this by considering the response of the masses when set in motion by an initial disturbance. For simplicity, set one of the masses as fixed so that the free-body diagram of the moving mass attached to the spring looks like Figure 1.2(b). We identify four forces acting on the mass displaced by an amount $u(t)$. The applied force $P_o(t)$ is the agent causing the displacement, the elastic force Ku attempts to return the mass to its original position, the inertia force $-M\ddot{u}$ acts so as to keep the mass doing whatever it is doing, and finally, the damping force $F^d = C\dot{u}$ attempts to slow down the motion. The balance of forces is

$$P_o - Ku - F^d = M\ddot{u} \quad \text{or} \quad M\ddot{u} + C\dot{u} + Ku = P(t) \tag{1.1}$$

This is the governing *equation of motion* (EoM); in this instance, it is a single, nonhomogeneous linear ordinary differential equation (ODE) with constant coefficients, but more generally, it is a system of nonlinear ODEs.

In addition to Newton's laws of motion, there are a couple of concepts that are very useful for our later developments, and these concern the various forms of work and energy. The *work* done by a force in moving a distance $d\hat{u}$ is given by the vector

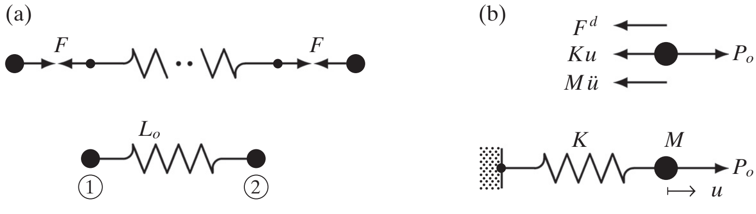

Figure 1.2. Simple spring-mass systems: (a) stretched spring showing equal (but opposite) forces on each mass; (b) the free-body diagram shows the retarding forces acting on the mass set in motion by the applied load P_o.

dot product

$$d\mathcal{W} = \hat{P}\cdot d\hat{r} = P_x du + P_y dv + P_z dw$$

Work is a scalar and has the units of force (N) times displacement (m) or N · m. This unit is given the special name *joule* (J), which is defined as the work done by 1 N moving through a distance of 1 m. Because work is a scalar, the work done by a system of forces is the simple scalar sum of the individual contributions. The work done by a force in stretching a spring between two states is

$$\Delta\,\mathcal{W} = \int_1^2 \hat{P}_o\cdot d\hat{u} = \int P_o\,du = \int Ku\,du = \frac{1}{2}K(u_2^2 - u_1^2) = \mathcal{U}_2 - \mathcal{U}_1$$

This form of work is called the *strain energy*, and we use the symbol $\mathcal{U} = 1/2Ku^2$. The work done on a particle during a finite time is

$$\Delta\,\mathcal{W} = \int_1^2 \hat{P}\cdot d\hat{u} = \int \frac{d(M\hat{v})}{dt}\cdot\frac{d\hat{u}}{dt}\,dt = \int M\hat{v}\cdot d\hat{v} = \frac{1}{2}M(v_2^2 - v_1^2) = \mathcal{T}_2 - \mathcal{T}_1$$

This form of work is called the *kinetic energy* of the particle, and we use the symbol $\mathcal{T} = 1/2Mv^2$. Although related to each other, work and energy are distinct concepts. We can say that forces perform work, but the system possesses energy. Therefore, when work is performed on a system, a change of energy occurs. When there are no dissipation mechanisms, energy is conserved, so

$$\mathcal{E} = \mathcal{U} + \mathcal{T} = \text{constant} \quad \text{or} \quad \frac{d\mathcal{E}}{dt} = \frac{d}{dt}\big(\mathcal{U} + \mathcal{T}\big) = 0 \tag{1.2}$$

This is useful for the simple single degree of freedom (SDoF) systems considered in this chapter but must be amended considerably for the complex systems of interest in the remaining chapters.

The concepts of work and energy form the foundation of our mechanics formulation in Chapter 2.

EXAMPLE 1.1. Establish the equation of motion for the simple pendulum shown in Figure 1.3.

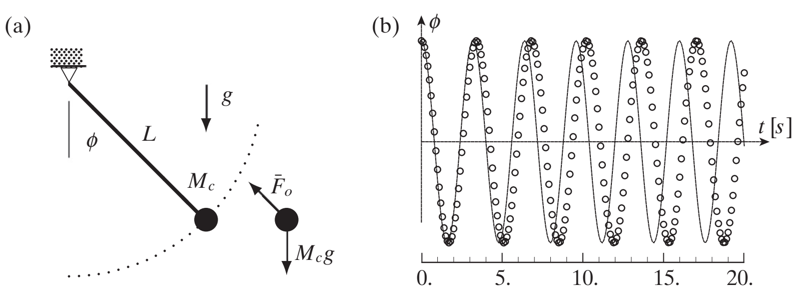

Figure 1.3. Simple pendulum system: (a) geometry and free body; (b) numerically generated time response.