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Fourier Analysis

Part I – Theory

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To my family

Contents

Preface

Fourier analysis is a central area of modern mathematics, comprising deep results that rely on advanced principles, as well as numerous aspects that require manipulative ingenuity. The power of the theory is illustrated by its wide applicability. Ideas originating in Fourier analysis permeate many essential developments of modern mathematics, bridging analysis with algebra and providing effective tools for an astonishing variety of applications. We list here an alphabetical sample of subjects, illustrating either areas of mathematics with strong links to Fourier analysis or real world applications of Fourier analysis, that are covered briefly in this textbook:[1] acoustics, complex analysis, functional analysis/operator theory, group theory/representation theory, heat flow, hydrodynamics, image processing, medical imaging, number theory, optics and astronomy, partial differential equations, probability and statistics, quantum mechanics, signal processing.

While formal approaches to Fourier analysis can be informative, to appreciate the subject fully and to strengthen the ability to use it in other contexts, one has to acquire a certain mathematical sophistication that draws on measure theory and functional analysis. Lebesgue's integral and the concepts of Hilbert and Banach spaces are intimately connected to Fourier analysis, providing not only an adequate setting but also being useful in obtaining fundamental results, often with surprisingly little effort. A detailed presentation of measure theory and functional analysis would be out of place in an introductory textbook, but

[1] The list is not exhaustive, being only indicative of the relevance of Fourier analysis to pure and applied mathematics. Part I of the textbook covers the theoretical background of Fourier analysis, while Part II is devoted to applications. Since it is impossible to draw a clear dividing line between Fourier theory and applications – for example, these aspects strongly overlap and intertwine in our discussion (in Part II) of the discrete Fourier transform and of the uncertainty principle – the difference between the material presented in the two volumes is more a question of emphasis.

ignoring these topics would amount to a lamentable attempt to run before we have learned to walk. For this reason we outline in Chapters 2 and 3 the principal facts about the Lebesgue integral and Hilbert/Banach spaces, as needed later, emphasising and illustrating the relevant conceptual ideas. This should provide some essential intuition that must, nevertheless, be adequately backed up by analytic rigour, so that we present at least sketchy proofs, avoiding only the proofs that demand an advanced degree of technical versatility. The reader may take on faith the results stated without proof, but detailed references for further study are provided. This material offers, to the interested reader, a basis for a solid grounding in these aspects and has been "class-tested" to groups of graduate students at Lund University and at the University of Vienna (during the academic years 2002–2004 and 2014–2015, respectively). However, the material in Chapters 2 and 3 is not an integral part of a standard course. Depending on the predilections of the lecturer, an all-encompassing or a minimalist point of view can be adopted. In the latter case, one can dispense altogether with measure-theoretic issues and work with the Lebesgue integral as if it were a Riemann integral, with the added bonus of Fatou's lemma and the monotone and dominated convergence theorems. The completeness, separability and density results for the spaces of integrable and square integrable functions can be taken for granted. Chapters 4–6 represent the core of the theory underpinning Fourier analysis, with various applications presented in Part II. Some applications are aimed at pure mathematicians, while others illustrate the relevance of Fourier analysis to physics and engineering. Each application was selected by virtue of its relevance and interest, but each is self-contained: the formulation of the problem is accessible and a full solution is presented. We avoid topics that can be covered only in part within a first course on Fourier analysis. The even distribution of pure and applied topics aims to cater to both mathematical backgrounds. Realistically, only about a third to a half of the applications presented in Part II can be covered in a lecture course. The available flexibility in the specific choice permits a suitable mix of pure and applied topics – the separation of pure and applied topics being, in the long run, detrimental to both areas. Whether the entire Chapter 6 belongs to the basic material on Fourier analysis is a matter of personal opinion, and thus open to debate. Parts of it could be viewed as optional reading material. Chapter 7 contains various selected advanced topics in Fourier analysis, illustrating some of the main directions in which the subject has developed. The material in Chapters 4–6, with the exception of the aspects related to distribution theory in Chapter 6, has been taught by the author as a one-semester course at King's College London during the academic years 2012–2014, while the distribution-theory aspects are an outgrowth of a lecture course on this topic at Trinity College

Dublin during 2008. In recent decades, Fourier analysis has known a period of intense technical and conceptual development which has led to a bewildering array of related topics. Nevertheless, there are a relatively small number of concepts that are commonly regarded as the bare essentials in the theory of Fourier analysis. A minimal list that could constitute a short, introductory course consists of: Section 1.1, Section 3.2, Sections 4.1–4.3, Sections 5.1–5.3, and Exercises 1.1, 3.3, 3.23, 4.1, 4.2, 4.4, 4.5, 4.11, 4.16, 4.19, 5.2, 5.5, 5.9, 5.10, 5.15, 5.16, 5.18. Each chapter is denoted by a numeral (for example, Chapter 5). The first section of the sixth chapter is denoted Section 6.1, and its second subsection is 6.1.2. Theorem 5.3 refers to the third theorem in Chapter 5 (without specification of the section or subsection), while Exercise 5.4 refers to the fourth exercise in Chapter 5. However, within Chapter 5, the 5 may be dropped and Exercise 4 used instead of Exercise 5.4.

The prerequisites are a thorough knowledge of advanced calculus and linear algebra. A large number of exercises are provided, ranging from easy to very hard, and these are supplied separately with hints and full solutions. The exercises are to be regarded as an integral part of the text, and the provided hints and solutions offer flexibility in calibrating the scale of the undertaking – in a tour it is better to admire the scenery at ease rather than to keep on schedule. Whenever the reader struggles with solving an exercise, it is worthwhile glancing at the hint prior to going through the solution that is available. Even if this proves insufficient, it might offer some valuable insight. We strive throughout for a somewhat detailed presentation, at the risk of boring those able to proceed faster. Such readers have the option of judicious skipping. We have tried to prevent our natural fondness for simplicity turning into an excuse to avoid difficulties at any price; quite often, certain difficulties are apparently circumvented rather than escaped altogether. Eventually, they may be encountered again, when they have multiplied, become more involved and hidden in a confusion of detail, which has been generated by lots of misdirected industry. On the other hand, we have tried to steer away from emphasising matters of pure technique – it is all too common to see failures of insight hidden under a blanket of excess technical detail, and focus on detail often leads to a narrowing of perspective. We hope to have found an acceptable balance between doing too little and attempting to do too much.

An appendix with some brief historical notes illustrates the international character of the underlying research efforts, being also indicative of the time needed for the crystallisation of specific concepts and ideas, as well as of their lasting value. There is a similarity between the struggle of early research mathematicians who developed and formalised a topic and the challenges embraced as one embarks upon the study of the topic.

In writing this textbook the author has acted primarily as a reporter, not a researcher: nearly all the results can be found in earlier books or in research publications. We try to offer a coherent exposition, arranging separate topics into a unified whole, and occasionally incorporating some recent developments. While we attempt to give credit where it is due, we also found that this is sometimes difficult or impossible and, as a result, in some instances, secondary sources have prevailed. The reading of parts of this book would be, we believe, beneficial during, or as a preparation for, a graduate school in mathematics – at least, the author wishes he had this material before beginning his own graduate studies.

I owe a debt of gratitude to Roger Astley of Cambridge University Press who encouraged this project from the beginning, being patient and understanding beyond the call of duty. I would like to thank the reviewers of an early and incomplete draft of the book for their constructive suggestions, which I have attempted to incorporate. I am grateful to several mathematicians for reading and commenting on the manuscript and for trying out parts of it on their classes. I cannot name one without naming them all, so they shall remain unnamed to avoid offence to those whose names have escaped me as I attempted to draw up a tentative list of acknowledgements, but they are all deeply appreciated. Despite their best efforts, there are very likely undetected errors that are my sole responsibility and for which I ask the reader to accept my apologies.