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Number Theory, Fourier Analysis and Geometric Discrepancy

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To Frances

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Introduction

Through this book we wish to achieve and connect the following three goals:

1) to present some elementary results in number theory;
2) to introduce classical and recent topics on the uniform distribution of infinite sequences and on the discrepancy of finite sequences in several variables;
3) to present a few results in Fourier analysis and use them to prove some of the theorems discussed in the two previous points.

The first part of this book is dedicated to the first goal. The reader will find some topics typically presented in introductory books on number theory: factorization, arithmetic functions and integer points, congruences and cryptography, quadratic reciprocity, and sums of two and four squares. Starting from the first few pages we introduce some simple and captivating findings, such as Chebyshev's theorem and the elementary results for the Gauss circle problem and for the Dirichlet divisor problem, which may lead the reader to a deeper study of number theory, particularly students who are interested in calculus and analysis.

In the second part we start with the uniformly distributed sequences, introduced in 1916 by Weyl and related to the strong law of large numbers and to Kronecker's approximation theorem. Then we introduce the definition of discrepancy, which is the quantitative counterpart of the uniform distribution and has natural applications in the computation of high-dimensional integrals. For the particular case of integer points we use different techniques to prove some classical but not trivial results for the Gauss circle problem and for the Dirichlet divisor problem. Then we introduce the *geometric discrepancy*, also known as *irregularities of distribution* because some of its main results show the existence of unavoidable errors in the approximation of a continuous object by a discrete sampling. This theory has grown over the last 60 years thanks to

the contributions by Roth, Schmidt, Beck and other authors, and it is presently a crossroads between number theory, combinatorics, Fourier analysis, algorithms and complexity, probability and numerical analysis [49]. Its current applications range from traditional science and engineering to modern computer science and financial mathematics [43].

A large number of the results in this book are proved through Fourier analytic arguments: pointwise convergence of Fourier series, completeness of the trigonometric system, trigonometric approximation, Poisson summation formula, exponential sums, decay of Fourier transforms and Bessel functions. The result is a short and self-contained course on Fourier analysis, which we present in parallel with the two previous points.

This book is based on a number theory course of 60–70 hours given by the author at the University of Milano-Bicocca for several years. It was a postgraduate course, but many undergraduate students attended it with success. The prerequisites are limited: no prior knowledge of number theory or Fourier analysis is necessary, we assume a bit of algebra and a solid background in calculus, we need the Lebesgue integral, but we use complex analysis only in the last chapter.

The lecture notes of the above course first appeared in the Italian book *Appunti su teoria dei numeri, analisi di Fourier e distribuzione di punti* [173]. We wish to thank the Unione Matematica Italiana who generously released the rights to this revised and expanded English version.

We are very grateful to the students who attended the course. We also wish to thank Anatoly Podkorytov, Giacomo Gigante, Leonardo Colzani, Luca Brandolini and William Chen, who have read the original draft. We thank William Chen also for permission and encouragement to freely use his notes [46] of the number theory courses he taught at Imperial College London. These notes were modified many times from notes used by various colleagues over many years, both at Imperial College London and University College London. It seems likely that Davenport is the original source of these notes.

We are also happy to thank all the people at Cambridge University Press who were involved in publishing this book, in particular Roger Astley, Roisin Munnelly, Samuel Harrison and Joanna Breeze.

Giancarlo Travaglini
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