NON-ASSOCIATIVE NORMED ALGEBRAS

Volume 2: Representation Theory and the Zel'manov Approach

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Non-Associative Normed Algebras

Volume 2: Representation Theory and the Zel'manov Approach

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To Ana María and Inés

Contents for Volume 2

Contents for Volume 1

Preface

The core of the book revisited

In the preface to Volume 1 we proposed as the 'leitmotiv' of our work to remove associativity in the abstract characterizations of unital (associative) C^*-algebras given either by the Gelfand–Naimark theorem or by the Vidav–Palmer theorem, and to study (possibly non-unital) closed $*$-subalgebras of the Gelfand–Naimark or Vidav–Palmer algebras born after removing associativity.

To be more precise, for a norm-unital complete normed (possibly non-associative) complex algebra A, we considered the following conditions:

(GN) (Gelfand–Naimark axiom). *There is a conjugate-linear vector space involution $*$ on A satisfying $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a\|^2$ for every a in A.*

(VP) (Vidav–Palmer axiom). $A = H(A,\mathbf{1}) + iH(A,\mathbf{1})$.

In both conditions, $\mathbf{1}$ denotes the unit of A, whereas, in (VP), $H(A,\mathbf{1})$ stands for the closed real subspace of A consisting of those elements $h \in A$ such that $f(h)$ belongs to $\mathbb{R}$ for every bounded linear functional f on A satisfying $\|f\| = f(\mathbf{1}) = 1$.

Contrary to what happens in the associative case [696, 725, 787, 930], in the non-associative setting, (GN) and (VP) are not equivalent conditions. Indeed, as proved in Lemma 2.2.5, it is easily seen that (GN) implies (VP), but, as shown by Example 2.3.65, the converse implication is not true. Therefore, after introducing 'alternative C^*-algebras' and 'non-commutative JB^*-algebras', and realizing that the former are particular cases of the latter, we specified how, by means of Theorems GN and VP which follow, the behaviour of the Gelfand–Naimark and the Vidav–Palmer axioms in the non-associative setting are clarified.

Theorem GN *Norm-unital complete normed complex algebras fulfilling the Gelfand–Naimark axiom are nothing other than unital alternative C^*-algebras.*

Theorem VP *Norm-unital complete normed complex algebras fulfilling the Vidav–Palmer axiom are nothing other than unital non-commutative JB^*-algebras.*

Then we announced as the main goal of our work to prove Theorems GN and VP, together with their unit-free variants, and to 'describe' alternative C^*-algebras and non-commutative JB^*-algebras by means of the so-called representation theory. Since Theorems GN and VP and the unit-free variant of Theorem GN were already proved in Theorems 3.2.5, 3.3.11, and 3.5.53, respectively, it remains the main objective of our work to prove the unit-free variant of Theorem VP, and to develop the representation theory of alternative C^*-algebras and non-commutative JB^*-algebras. We now do this in Chapters 5 and 6 respectively. Indeed, the unit-free variant of Theorem VP is proved in Theorem 5.9.9, whereas the representation theory of alternative C^*-algebras and non-commutative JB^*-algebras can be summarized by means of Corollaries 6.1.11 and 6.1.12, Theorem 6.1.112, and Corollary 6.1.115.

The content of Volume 2

As we commented in the preface of Volume 1, the dividing line between the two volumes could be drawn between what can be done before and after involving the holomorphic theory of JB^*-triples and the structure theory of non-commutative JB^*-algebras. Then the content of Volume 1 was described in some detail, and a tentative content of Volume 2 was outlined. Now we are going to specify with more precision the content of the present second volume.

Chapter 5

The main goal of this first chapter of Volume 2 is to prove what can be seen as a unit-free version of the non-associative Vidav–Palmer theorem, namely that *non-commutative JB^*-algebras are precisely those complete normed complex algebras having an approximate unit bounded by one, and whose open unit ball is a homogeneous domain* [365] (see Theorem 5.9.9). Some ingredients in the long proof of this result were already established in Volume 1. This is the case of the Bohnenblust–Karlin Corollary 2.1.13, the non-associative Vidav–Palmer theorem proved in Theorem 3.3.11 as well as its dual version shown in Corollary 3.3.26, Proposition 3.5.23 (that every non-commutative JB^*-algebra has an approximate unit bounded by one), Theorem 4.1.45 (that non-commutative JB^*-algebras are JB^*-triples in a natural way), and the equivalence (ii)$\Leftrightarrow$(vii) in the Braun–Kaup–Upmeier Theorem 4.2.24.

♣ The new relevant ingredients which are proved in the chapter are the following:

(i) Edward's fundamental Fact 5.1.42, which describes how JBW-algebras and JBW^*-algebras are mutually determined, and implies, via [738], the uniqueness of the predual of any non-commutative JBW^*-algebra (see Theorem 5.1.29(iv)).
(ii) The Kaup–Stachó contractive projection theorem for JB^*-triples (see Theorem 5.6.59).
(iii) Kaup's holomorphic characterization of JB^*-triples as those complex Banach spaces whose open unit ball is a homogeneous domain (see Theorem 5.6.68).

(iv) Dineen's celebrated result that the bidual of a JB^*-triple is a JB^*-triple (see Proposition 5.7.10).
(v) The Barton–Horn–Timoney basic theory of JBW^*-triples establishing the separate w^*-continuity of the triple product of a given JBW^*-triple (see Theorem 5.7.20) and the uniqueness of the predual (see Theorem 5.7.38).
(vi) The Barton–Timoney theorem that the predual of any JBW^*-triple is L-embedded (see Theorem 5.7.36).
(vii) The Chu–Iochum–Loupias result that bounded linear operators from a JB^*-triple to its dual are weakly compact (see Corollary 5.8.33) or, equivalently, that all continuous products on the Banach space of a JB^*-triple are Arens regular (see Fact 5.8.39).

The original references for the results just listed are [222], [382, 597], [381], [213], [854, 979], [854], and [172], respectively. Our proof of these results are not always the original ones, although sometimes the latter underlie the former. This is the case of results (ii) and (iii), which in our development depend on the foundations of the infinite-dimensional holomorphy done in [710, 751, 814, 837, 1113, 1114, 1124] (see Sections 5.2 to 5.6), on the design of proof suggested in [710, Section 2.5], and, at the end, on numerical range techniques included in Subsection 5.6.3. On the other hand, our proof of result (v) is new, and, contrary to what happens in the original one, it avoids any Banach space result on uniqueness of preduals. Indeed, our proof of Theorem 5.7.20 involves only result (ii) and the Barton–Timoney Theorem 5.7.18, whereas our proof of Theorem 5.7.38 depends only on Theorem 5.7.20 (whose proof has been just remarked on), result (i), and Horn's Corollary 5.7.28(i)(b).

Concerning result (vii), it is noteworthy that a much finer theorem is proved in [172]. Namely, that every bounded linear operator from a JB^*-triple to its dual factors through a complex Hilbert space. The proof of this more general theorem (a sketch of which can be found in §5.10.151) is very involved, and shall not be completely discussed in our work. As a matter of fact, we re-encounter result (vii) by combining results (iv) and (vi) with Corollary 5.8.19 (asserting that, if Y is a Banach space such that Y' has property (V^*), then every bounded linear operator from Y to Y' is weakly compact) and Theorem 5.8.27 (that L-embedded Banach spaces have property (V^*)). Corollary 5.8.19 and Theorem 5.8.27 just reviewed are due to Godefroy–Iochum [957] and Pfitzner [1044], respectively. Nevertheless, the proof of Corollary 5.8.19 in [957] relies heavily on Proposition 5.8.14, whose arguments have been lost in the literature (see §5.8.42). Our proof of Proposition 5.8.14 is taken from Pfitzner's private communication [1047].

Once the main objective of the chapter is reached in Section 5.9, the chapter concludes with a section devoted to some complements on non-commutative JB^*-algebras and JB^*-triples.

In Subsection 5.10.1 we introduce the strong* topology of a non-commutative JBW^*-algebra [19] and apply it to build up a functional calculus at each normal element a of a non-commutative JBW^*-algebra A, which extends the continuous

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functional calculus (cf. Corollary 4.1.72) and has a sense for all real-valued bounded lower semicontinuous functions on J-sp(A,a). Then we follow [366] to prove a variant for non-commutative JBW^*-algebras of Kadison's isometry theorem for unital C^*-algebras (cf. Theorem 2.2.29), a consequence of which is that linearly isometric non-commutative JBW^*-algebras are Jordan-$*$-isomorphic. (We recall that linearly isometric (possibly non-unital) C^*-algebras are Jordan-$*$-isomorphic (a consequence of Theorem 2.2.19), but that linearly isometric (even unital) non-commutative JB^*-algebras need not be Jordan-$*$-isomorphic (cf. Antitheorem 3.4.34).) We also prove the generalization to non-commutative JBW^*-algebras of Akemann's theorem [826] asserting the coincidence of the strong* and Mackey topologies on bounded subsets of any W^*-algebra.

In Subsection 5.10.2 we introduce and study the strong* topology of a JBW^*-triple as done by Barton and Friedman [853, 60], and follow [1061] to prove that, when a non-commutative JBW^*-algebra is viewed as a JBW^*-triple, its new (triple) strong* topology coincides with the (algebra) strong* topology introduced in Subsection 5.10.1. We also prove Zizler's refinement [1137] of Lindenstrauss's theorem [1001] on norm-density of operators whose transpose attain their norm, and apply it to prove a variant for JBW^*-triples of the so-called little Grothendieck's theorem [853, 964, 1040, 1052].

In Subsection 5.10.3 we provide the reader with a full non-associative discussion of the Kadison–Paterson–Sinclair Theorem 2.2.19 on surjective linear isometries of (possibly non-unital) C^*-algebras [366]. To this end we introduce the multiplier non-commutative JB^*-algebra $M(A)$ of a given non-commutative JB^*-algebra A, and prove that $M(A)$ coincides with the JB^*-triple of multipliers [873] of the JB^*-triple underlying A. Then we also prove that the Kadison–Paterson–Sinclair theorem remains true verbatim for surjective linear isometries from non-commutative JB^*-algebras to alternative C^*-algebras, and that no further verbatim generalization is possible.

Chapter 6

Implicitly, the representation theory of JB-algebras underlies our work since, without providing the reader with a proof, we took from the Hanche-Olsen–Stormer book [738] the very deep fact that the closed subalgebra of a JB-algebra generated by two elements is a JC-algebra (cf. Proposition 3.1.3). In that way we were able to develop the basic theory of non-commutative JB^*-algebras (including the non-associative Vidav–Palmer Theorem 3.3.11 and Wright's fundamental Fact 3.4.9 which describes how JB-algebras and JB^*-algebras are mutually determined) without any further implicit or explicit reference to representation theory. In fact, we avoided any dependence on representation theory throughout all of Volume 1, and to the end of Chapter 5 of the present volume.

Now, in Chapter 6, we conclude the basic theory of non-commutative JB^*-algebras, and follow [19, 124, 125, 222, 481, 482, 641] to develop in depth the representation theory of non-commutative JB^*-algebras and, in particular, that of alternative C^*-algebras. To this end, in Subsection 6.1.1 we introduce non-commutative JBW^*-factors and non-commutative JBW^*-factor representations of a given non-commutative JB^*-algebra, and prove that every non-commutative JB^*-algebra has a faithful family of type I non-commutative JBW^*-factor representations. When these results specialize for classical C^*-algebras, type I non-commutative JBW^*-factors are nothing other than the (associative) W^*-factors consisting of all bounded linear operators on some complex Hilbert space [738, Proposition 7.5.2], and, consequently, type I W^*-factor representations of a C^*-algebra A are precisely the irreducible representations of A on complex Hilbert spaces. Subsection 6.1.2 deals with a first application of the representation theory outlined above, which allows us to show that non-commutative JB^*-algebras are associative and commutative if (and only if) they have no nonzero nilpotent element. As a consequence, we obtain that alternative C^*-algebras are commutative if and only if they have no nonzero nilpotent element [340]. This generalizes Kaplansky's associative forerunner [761, Theorem B in Appendix III]. In Subsection 6.1.3, we involve the theory of JB-algebras [738], and invoke result (i) in ♣ to classify all (commutative) JBW^*-factors. This classification is applied to prove that i-special JB^*-algebras are JC^*-algebras. In Subsection 6.1.4, we combine the result just reviewed with Zel'manovian techniques [437, 662] to prove that, if J is a prime JB^*-algebra, and if J is neither quadratic (cf. Corollary 3.5.7) nor equal to the unique JB^*-algebra whose self-adjoint part is $H_3(\mathbb{O})$ (cf. Example 3.1.56 and Theorem 3.4.8), then either there exists a prime C^*-algebra A such that J is a closed $*$-subalgebra of the JB^*-algebra $M(A)^{\mathrm{sym}}$ containing A, or there exists a prime C^*-algebra A with a $*$-involution τ such that J is a closed $*$-subalgebra of $M(A)^{\mathrm{sym}}$ contained in $H(M(A),\tau)$ and containing $H(A,\tau)$ [255]. In Subsection 6.1.5, we introduce totally prime normed algebras and ultraprime normed algebras, and prove that totally prime normed complex algebras are centrally closed, and that ultraprime normed algebras are totally prime [149]. Then we combine the classification theorem of prime JB^*-algebras reviewed above with the fact that prime C^*-algebras are ultraprime [1012] to show that prime non-commutative JB^*-algebras are ultraprime, and hence centrally closed. In Subsection 6.1.6, we combine the central closedness of prime non-commutative JB^*-algebras with a topological reading of McCrimmon's paper [436] to prove that non-commutative JBW^*-factors are either commutative or simple quadratic or of the form $B^{(\lambda)}$ for some (associative) W^*-factor B and some $0 \leq \lambda \leq 1$. This theorem is originally due to Braun [124]. As a consequence, alternative W^*-factors are either associative or equal to the alternative C^*-algebra of complex octonions (cf. Proposition 2.6.8).

Now that we have reviewed Section 6.1 in detail, we will explain the content of the remaining sections of Chapter 6. Section 6.2 deals with the main applications of the representation theory, namely the structure of alternative C^*-algebras [125, 331, 481], the definition and properties of the strong topology of a non-commutative JBW^*-algebra [482], and the classification of prime non-commutative JB^*-algebras

[363]. Finally, Section 6.3 deals with a rather incidental application. Indeed, we follow [860] to prove a Le Page type theorem for non-commutative JB^*-algebras, and discuss Le Page's theorem [999] in a general non-associative and non-star setting.

Chapter 7

This chapter deals with the analytic treatment of Zel'manov's prime theorems for Jordan structures, thus continuing the approach begun in Subsection 6.1.4.

In Subsection 7.1 we follow [448, 449] to prove as the main result that, if X is a prime JB^*-triple which is neither an exceptional Cartan factor nor a spin triple factor, then either there exist a prime C^*-algebra A and a self-adjoint idempotent e in the C^*-algebra $M(A)$ of multipliers of A such that X is a closed subtriple of $M(A)$ contained in $eM(A)(\mathbf{1}-e)$ and containing $eA(\mathbf{1}-e)$, or there exist a prime C^*-algebra A, a self-adjoint idempotent $e \in M(A)$, and a $*$-involution τ on A with $e+e^\tau = \mathbf{1}$ such that X is a closed subtriple of $M(A)$ contained in $H(eM(A)e^\tau, \tau)$ and containing $H(eAe^\tau, \tau)$.

Among the many tools involved in the proof of the above classification theorem, we emphasize Horn's description of Cartan factors [330], the core of the proof of Zel'manov's prime theorem for Jordan triples [663, 1133, 1134], and the complementary work by D'Amour and McCrimmon on the topic [920, 921]. Proofs of these tools are not discussed in our development. The main results in the Friedman–Russo paper [270], whose proofs are outlined in our development, are also involved. It is noteworthy that, through the description of prime JB^*-algebras proved in Subsection 6.1.4, Zel'manov's work underlies again the proof of the classification theorem of prime JB^*-triples we are dealing with.

In Section 7.2 we survey in detail other applications of Zel'manov's prime theorems on Jordan structures to the study of normed Jordan algebras and triples.

In Subsection 7.2.1 we include the general complete normed version [146] of the Anquela–Montaner–Cortés–Skosyrskii classification theorem of J-primitive Jordan algebras [21, 585], as well as the more precise classification theorem of J-primitive JB^*-algebras [255, 525].

In Subsection 7.2.2 we include structure theorems for simple normed Jordan algebras [151] (see also [539]) and non-degenerately ultraprime complete normed Jordan complex algebras [152] (see also [428, 855]). This subsection deals also with the limits of normed versions of Zel'manov prime theorems, a question which was first considered in [893], and culminates in the paper of Moreno, Zel'manov, and the authors [147] where it is proved that an associative polynomial $\mathbf{p}$ over $\mathbb{K}$ is a Jordan polynomial if and only if, for every algebra norm $\|\cdot\|$ on the Jordan algebra $M_\infty(\mathbb{K})^{\mathrm{sym}}$, the action of $\mathbf{p}$ on $M_\infty(\mathbb{K})$ is $\|\cdot\|$-continuous (see also [447, 1082]).

Subsection 7.2.3 deals with the so-called norm extension problem, which in its roots is crucially related to normed versions of Zel'manov's prime theorems. The first significative progress on this problem (reviewed of course in this subsection) is due to Rodríguez, Slinko, and Zel'manov [538], who as the main result prove that, if A is a real or complex associative algebra with linear algebra involution $*$, if A is a '$*$-tight envelope of $H(A,*)$', if the Jordan algebra $H(A,*)$ is semiprime, and if $\|\cdot\|$ is a complete algebra norm on $H(A,*)$, then there exists an algebra norm on A whose restriction to $H(A,*)$ is equivalent to $\|\cdot\|$. The appropriate versions for Jordan triples of the results of [538], due to Moreno [1025, 1026], are also included. The subsection concludes with a full discussion of results on the norm extension problem in a general non-associative setting. The main reference for this topic is [1029]. Other related results in [1027, 1059, 1064] are also reviewed.

Chapter 8

We devote this concluding chapter to developing some of our favourite parcels of the theory of non-associative normed algebras, not previously included in our work.

The first section of the chapter deals with H^*-algebras, incidentally introduced in Volume 1 of our work. The reasonably well-behaved co-existence of two structures, namely that of an algebra and that of a Hilbert space, becomes the essence of semi-H^*-algebras. Indeed, they are complete normed algebras A endowed with a (vector space) conjugate-linear involution $*$, and whose norm derives from an inner product in such a way that, for each $a \in A$, the adjoint of the left multiplication L_a is precisely L_{a^*}, and the adjoint of the right multiplication R_a is R_{a^*}. Since Ambrose's pioneering paper [20], it is well-known that associative semi-H^*-algebras with zero annihilator are H^*-algebras, i.e. their involutions are algebra involutions. But this is no longer true in general.

We begin Subsection 8.1.1 by recalling those results on semi-H^*-algebras, which were already proved in Volume 1 of our work. Then we introduce the classical topologically simple associative complex H^*-algebra $\mathscr{HS}(H)$ of all Hilbert–Schmidt operators on a nonzero complex Hilbert space H, and show how this algebra allows us to construct natural examples of Jordan and Lie H^*-algebras. After showing how the norm of a semi-H^*-algebra with zero annihilator determines its involution, we prove that power-associative H^*-algebras are non-commutative Jordan algebras [714].

In Subsection 8.1.2, we establish two fundamental structure theorems for a semi-H^*-algebra A, which, in two successive steps, reduce the general case to the one that A has zero annihilator, and the case that A has zero annihilator to the one that A is topologically simple [199].

According to the structure theory commented in the preceding paragraph, topologically simple semi-H^*-algebras merit being studied in depth. This is done in Subsection 8.1.3. To this end we introduce totally multiplicatively prime normed algebras, show that they are totally prime, and prove that topologically simple complex semi-H^*-algebras are totally multiplicatively prime [889]. Since, as we already commented in our review of Subsection 6.1.5, totally prime normed complex algebras are centrally closed, it follows that topologically simple complex H^*-algebras are centrally closed [148, 149].

The central closedness of topologically simple complex H^*-algebras just reviewed becomes the key tool of Subsection 8.1.4, where we prove that derivations of complex semi-H^*-algebras with zero annihilator are continuous [624], and that dense-range algebra homomorphisms from complete normed complex algebras to complex H^*-algebras with zero annihilator are also continuous [526].

In Subsection 8.1.5 we show that isomorphic complex H^*-algebras with zero annihilator are $*$-isomorphic, and that bijective algebra $*$-homomorphisms between topologically simple complex H^*-algebras are positive multiples of isometries (hence, essentially, a topologically simple complex H^*-algebra has a unique H^*-algebra structure) [198]. These results follow from a structure theorem for bijective algebra homomorphisms between complex H^*-algebras with zero annihilator, which becomes the appropriate H^*-variant of the structure theorem for bijective algebra homomorphisms between non-commutative JB^*-algebras proved in Theorem 3.4.75.

In Subsection 8.1.6, we prove the appropriate H^*-variant of the Jordan characterization of C^*-algebras established in Theorem 3.6.30 [518]. A more than satisfactory H^*-variant of Theorem 3.6.25 is also obtained [624].

Subsection 8.1.7 is devoted to providing us with the appropriate tools to transfer results from complex semi-H^*-algebras to real ones. The basic tool asserts that the complexification of any real (semi-)H^*-algebra becomes a complex (semi-)H^*-algebra in a natural way. This quite elementary fact already allows to convert many complex results into real ones, all of them involving the assumption that the algebra has zero annihilator. The treatment of topologically simple real (semi-)H^*-algebras is more elaborated: there are no topologically simple real (semi-)H^*-algebras other than topologically simple complex (semi-)H^*-algebras, regarded as real algebras, and the real (semi-)H^*-algebras of all fixed points for an involutive conjugate-linear algebra $*$-homomorphism on a topologically simple complex (semi-)H^*-algebra [142]. This reduction of topologically simple real (semi-) H^*-algebras to complex ones allows us to transfer the remaining results known in the complex setting to the real setting. In particular, we prove that dense-range algebra homomorphisms from H^*-algebras with zero annihilator to topologically simple H^*-algebras are surjective. Then, after introducing H^*-ideals of an arbitrary normed $*$-algebra, we prove that topologically simple normed $*$-algebras have at most one H^*-ideal [687].

We begin Subsection 8.1.8 by introducing the complete normed complex $*$-algebra $(\mathscr{TC}(H), \|\cdot\|_\tau)$ of all trace-class operators on a complex Hilbert space H, as well as the $\|\cdot\|_\tau$-continuous trace-form on it. Then we show that $(\mathscr{TC}(H), \|\cdot\|_\tau)$ can be intrinsically determined into the H^*-algebra $(\mathscr{HS}(H), \|\cdot\|)$ of all Hilbert–Schmidt operators on H. This fact allows us to replace $\mathscr{HS}(H)$ with an arbitrary real or complex (possibly non-associative) semi-H^*-algebra A with zero annihilator, to build an appropriate substitute of $(\mathscr{TC}(H), \|\cdot\|_\tau)$ into A, denoted by $(\tau c(A), \|\cdot\|_\tau)$, and to discuss whether or not a $\|\cdot\|_\tau$-continuous trace-form on $\tau c(A)$ does exist. We prove that, for a semi-H^*-algebra A with zero annihilator, $\tau c(A)$ is a $*$-invariant ideal of A, $(\tau c(A), \|\cdot\|_\tau)$ is both a normed algebra and a dual Banach space, and the existence of a $\|\cdot\|_\tau$-continuous trace-form on $\tau c(A)$ depends on the existence of

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an 'operator-bounded' approximate unit in A [424]. This, together with deep results established in Volume 1 (namely Theorem 3.5.53 and Proposition 4.5.36(ii)), allows us to prove that a complex H^*-algebra A with zero annihilator is alternative if and only if $(A, \|\cdot\|)$ has an approximate unit operator-bounded by 1, and the predual of $(\tau c(A), \|\cdot\|_\tau)$ is a non-associative C^*-algebra.

In the concluding Subsection 8.1.9, we survey the classification theorems of topologically simple H^*-algebras in the most familiar classes of algebras. Thus, starting from the well-known fact that there are no topologically simple associative complex H^*-algebras other than those of the form $\mathscr{HS}(H)$ for a nonzero complex Hilbert space H [20, 374], the corresponding theorems for topologically simple alternative [1042], Jordan and non-commutative Jordan [199, 1118, 1119], Lie [197, 460, 687], Malcev [141], or structurable [140, 144] H^*-algebras are established.

Section 8.2 deals with generalized annihilator normed algebras, which become non-star generalizations of H^*-algebras with zero annihilator. We prove that any generalized annihilator complete normed real or complex algebra with zero weak radical (cf. Definition 4.4.39) is the closure of the direct sum of its minimal closed ideals, which are indeed topologically simple normed algebras [259]. We also show that the weak radical of any real or complex semi-H^*-algebra coincides with its annihilator, so that the structure theorem for semi-H^*-algebras with zero annihilator proved in Subsection 8.1.2 is rediscovered. We introduce multiplicatively semiprime algebras (i.e. algebras such that both they and their multiplication algebras are semiprime), and show that generalized annihilator normed algebras are multiplicatively semiprime [876]. Even more, we characterize generalized annihilator normed algebras among those normed algebras which are multiplicatively semiprime. We introduce generalized complemented normed algebras, which are particular cases of generalized annihilator normed algebras, and prove that, if A is a generalized complemented complete normed algebra with zero weak radical, and if $\{A_i\}_{i\in I}$ stands for the family of its minimal closed ideals, then for each $a \in A$ there exists a unique summable family $\{a_i\}_{i\in I}$ in A such that $a_i \in A_i$ for every $i \in I$, and $a = \sum_{i\in I} a_i$ [259, 846].

Section 8.3 deals with other complements into the theory of non-associative normed algebras. In Subsection 8.3.1 we prove that algebra homomorphisms from complete normed complex algebras to complete normed complex algebras with no nonzero two-sided topological divisor of zero are continuous [529]. In Subsection 8.3.2 we show that complete normed J-semisimple non-commutative Jordan complex algebras, each element of which has a finite J-spectrum, are a finite direct sum of closed simple ideals which are either finite-dimensional or quadratic, and derive that complete normed semisimple alternative complex algebras, each element of which has a finite spectrum, are finite-dimensional [91]. After the usual subsection devoted to historical notes and comments, we include a comprehensive survey on the more significant results on normed Jordan algebras which have been not previously developed in our work.

The concluding Section 8.4 deals with the non-associative discussion done in [452, 453] of the Rota–Strang paper [544] (so in particular of Proposition 4.5.2, cf. p. 632 of Volume 1), and of the theory of topologically nilpotent normed (associative) algebras developed in [927, 928, 929, 1020] (see also [786, pp. 515–7], [1156, Section 11], and §8.4.121). The section discusses also non-associative versions of related results published in [569, 615, 1083] (see also [1030]), and incorporates proofs of most auxiliary results invoked but not proved in [452, 453]. Among these proofs, we emphasize that of Theorem 8.4.76, courtesy of Shulman and Turovskii.

In Subsection 8.4.1 we introduce the notion of (joint) spectral radius $\mathfrak{r}(S)$ of a bounded subset S of any normed algebra A. Then we prove one of the key results in the whole section, namely that, *if A is a normed algebra, and if S is a bounded subset of A with* $\mathfrak{r}(S) < 1$*, then the multiplicatively closed subset of A generated by S is bounded, and has the same spectral radius as S.*

In Subsection 8.4.2, we introduce topologically nilpotent normed algebras as those normed algebras whose closed unit balls have zero spectral radius. Among the results obtained, we emphasize the following:

(i) A normed associative algebra A is topologically nilpotent if and only if so is the normed Jordan algebra A^{sym} obtained by symmetrization of its product.
(ii) Every non-topologically nilpotent normed algebra can be equivalently algebra-renormed in such a way that the spectral radius of the corresponding closed unit ball is arbitrarily close to 1.
(iii) Every topologically nilpotent complete normed algebra is equal to its weak radical.

In Subsection 8.4.3, we show that, for every member A in a large class of normed algebras (which contains all commutative C^*-algebras, all JB-algebras, and all absolute-valued algebras), the conclusion in Proposition 4.5.2 has the following stronger form: for each bounded and multiplicatively closed subset S of A we have that $\sup\{\|s\| : s \in S\} \leq 1$.

In Subsection 8.4.4, we involve in our development tensor products of algebras. Thus we prove that the projective tensor product of two normed algebras is topologically nilpotent whenever some of them are topologically nilpotent, and that in fact the converse is true whenever some of them are associative. Moreover, associativity in the above converse cannot be removed. We also prove that a normed algebra A is topologically nilpotent if and only if so is the normed algebra $C_0(E,A)$ for some (equivalently, every) Hausdorff locally compact topological space E. The results obtained about tensor products of normed algebras are then applied to show that most notions introduced in the section can be non-trivially exemplified into a class of algebras almost arbitrarily prefixed.

On the historical notes

As in Volume 1, each section of the present volume concludes with a subsection devoted to historical notes and comments. Paraphrasing Dinnen [1155, p. X], in these notes ‘we provide information on the history of the subject and references for the material presented. We have tried to be as careful as possible in this regard and take responsibility for the inevitable errors. Accurate and comprehensive records of this kind are not a luxury but essential background information in appreciating and understanding a subject and its evolution’.

Errata: A list of errata for Volume 1 can be found in the web page of Volume 2: www.cambridge.org/9781107043114. We hope to continue this for both Volumes. Please send corrections to: cabrera@ugr.es and/or apalacio@ugr.es.

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