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Non-commutative JBW^*-algebras, JB^*-triples revisited, and a unit-free Vidav–Palmer type non-associative theorem

Non-commutative JB^*-algebras (cf. Definition 3.3.1) have become central objects in the first volume of our work since, in the unital case, they are the solution to the general non-associative Vidav–Palmer theorem (see Theorem 3.3.11), aned contain alternative C^*-algebras (cf. §2.3.62) which in their turn become the solution to the general non-associative Gelfand–Naimark theorem (see Theorem 3.5.53). As a concluding main result in the present chapter, we will prove a general non-associative characterization of non-commutative JB^*-algebras (see Theorem 5.9.9), a germ of which could be the following.

Fact 5.0.1 *A norm-unital complete normed complex algebra is a non-commutative JB^*-algebra (for some involution) if and only if it is linearly isometric to a JB^*-triple.*

The proof, which only involves results established in the first volume of our work, goes as follows.

Proof The 'only if' part follows from Theorem 4.1.45. To prove the 'if' part, let us recall that, given a complex normed space X and a norm-one element $u \in X$, $H(X,u)$ denotes the set of all hermitian elements of X relative to u (cf. Definition 2.1.12). Now let A be a norm-unital complete normed complex algebra such that there exists a linear isometry ϕ from A onto some JB^*-triple J. Then, by Corollary 2.1.13, $\phi(\mathbf{1})$ is a vertex of $\mathbb{B}_J$, and hence, by the implication (vii)$\Rightarrow$(ii) in Theorem 4.2.24, J is the underlying Banach space of a JB^*-algebra with unit $\phi(\mathbf{1})$. Therefore, by Lemma 2.2.8(iii), we have

$$J = H(J,\phi(\mathbf{1})) + iH(J,\phi(\mathbf{1})),$$

and hence

$$A = H(A,\mathbf{1}) + iH(A,\mathbf{1}).$$

Now, by the non-associative Vidav–Palmer theorem (cf. Theorem 3.3.11), A is a non-commutative JB^*-algebra. □

The actual formulation of Theorem 5.9.9 avoids 'norm-unital', and replaces 'is linearly isometric to a JB^*-triple' with 'has an approximate unit bounded by one and its open unit ball is a homogeneous domain (see Definition 5.3.53)'. To reach Theorem 5.9.9 from Fact 5.0.1, we must go a long way, with most steps having their own interest. Thus in Section 5.1 we introduce non-commutative JBW^*-algebras (i.e. non-commutative JB^*-algebras which are dual Banach spaces), and prove Edwards' results [222] relating non-commutative JBW^*-algebras with JBW-algebras (i.e. JB-algebras which are dual Banach spaces). In Sections 5.2, 5.3, 5.4, and 5.5 we study in detail the algebraic and analytic structure of the sets of all biholomorphic automorphisms and of all complete holomorphic vector fields on a bounded domain in a complex Banach space. This study culminates in Section 5.6, where we prove Kaup's characterization of JB^*-triples [380, 381] as those complex Banach spaces the open unit balls of which are homogeneous domains. Sections 5.7 and 5.8 are devoted to establishing the basic theory of JBW^*-triples (i.e. JB^*-triples which are dual Banach spaces) and of operators into the predual of a JBW^*-triple. These sections contain relevant results originally due Dineen [213], Barton–Timoney [854], Horn [979], and Chu–Iochum–Loupias [172]. It is noteworthy that our proofs of the Barton–Horn–Timoney Theorems 5.7.20 and 5.7.38 (asserting the separate w^*-continuity of the product and the uniqueness of the predual of a JBW^*-triple) are new and avoid any Banach space result on uniqueness of preduals. On the other hand, one of the crucial steps in our proof of the Chu–Iochum–Loupias Theorem 5.8.32 (asserting that bounded linear operators from a JB^*-triple to the predual of a JBW^*-triple are weakly compact) consists of Proposition 5.8.14, a result whose proof is difficult to find in the literature. We include a complete and self-contained proof of this result, which has been communicated to us by Pfitzner [1047]. Section 5.9 contains the (conclusion of) proof of the commented refinement of Fact 5.0.1, namely that non-commutative JB^*-algebras are precisely those complete normed complex algebras having a bounded approximate unit and whose open unit ball is a homogeneous domain.

The chapter concludes with Section 5.10, which contains different complements on non-commutative JB^*-algebras and JB^*-triples. Indeed, we study in deep the strong* topology of a non-commutative JBW^*-algebra and of a JBW^*-triple, as well as linear isometries between non-commutative JB^*-algebras.

5.1 Non-commutative JBW^*-algebras

Introduction This section is devoted to establishing the basic theory of non-commutative JBW^*-algebras. As main results we prove Edwards' theorems [222] asserting that a non-commutative JB^*-algebra A is a non-commutative JBW^*-algebra if and only if its self-adjoint part $H(A,*)$ is a JBW-algebra, and that, if this is the case, then the predual of A is unique, the involution of A is w^*-continuous, and the product of A is separately w^*-continuous (see Theorems 5.1.29 and 5.1.38 and Corollary 5.1.30). The proof of the uniqueness of the predual involves deep results of the theory of JB-algebras (see Theorem 5.1.27) which, as we did in the first volume of our work in similar occasions, are taken from [738] without proof. On the other hand, the proof of the separate w^*- continuity of the product in the non-commutative case follows an argument in [481].

The section also contains theorems taken from [481] asserting that, in a non-commutative JB^*-algebra, M-ideals are precisely the closed ideals (Theorem 5.1.22(i)) and that the predual of a non-commutative JBW^*-algebra is an L-summand of the dual (Theorem 5.1.32). The section concludes by revisiting real non-commutative JB^*-algebras in order to prove that c_0 is an M-ideal of its bidual (Corollary 5.1.57), a result which will be needed later in the proof of Theorem 5.8.27.

5.1.1 The results

Lemma 5.1.1 *Let A be an algebra over $\mathbb{K}$ with zero annihilator, and let I,J be ideals of A such that $A = I \oplus J$. Then $I = \{a \in A : aJ = Ja = 0\}$.*

Proof Put $K := \{a \in A : aJ = Ja = 0\}$. The inclusion $I \subseteq K$ is clear since

$$IJ + JI \subseteq I \cap J = 0.$$

Conversely, let a be in K, and write $a = x + y$ with $x \in I$ and $y \in J$. Then, by the inclusion just proved, $y = a - x \in K \cap J$. But, since $J \subseteq \{a \in A : aI = Ia = 0\}$, we derive that $y \in \text{Ann}(A) = 0$. Therefore $a = x \in I$. □

By a *direct summand* of an algebra A over $\mathbb{K}$ we mean any ideal I of A such that there exists another ideal J of A satisfying $A = I \oplus J$.

As a straightforward consequence of Lemma 5.1.1, we get the following.

Fact 5.1.2 *Let A be a normed algebra over $\mathbb{K}$ with zero annihilator, and let I be a direct summand of A. Then I is closed in A.*

Definition 5.1.3 Let X be a normed space over $\mathbb{K}$.

(i) By an *M-projection* on X we mean a linear projection $P : X \to X$ such that

$$\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\} \text{ for every } x \in X.$$

(ii) A subspace Y of X is said to be an *M-summand* of X if Y is the range of an M-projection on X.

Lemma 5.1.4 *Let A be a JB^*-algebra, and let I be a direct summand of A. Then, I is an M-summand of* (the Banach space underlying) A.

Proof Let J be an ideal of A such that $A = I \oplus J$. By Fact 5.1.2, I and J are closed in A, and then, by Proposition 3.4.13, they are $*$-invariant. Therefore, I and J are new JB^*-algebras, so $I \times J$ is a JB^*-algebra under the sup norm, and the mapping $(x, y) \to x + y$ from $I \times J$ to A is a bijective algebra $*$-homomorphism. It follows from Proposition 3.4.4 that $\|x + y\| = \max\{\|x\|, \|y\|\}$ for every $(x, y) \in I \times J$. Thus the projection from A onto I corresponding to the decomposition $A = I \oplus J$ becomes an M-projection. □

§5.1.5 As usual, by a *dual Banach space over* $\mathbb{K}$ we mean (the first component of) a couple (X,Y), where X and Y are Banach spaces over $\mathbb{K}$ such that $Y' = X$. The Banach space Y is called the *predual* of X, and is usually denoted by X_*.

Definition 5.1.6 By a *non-commutative JBW^*-algebra* (respectively, a *JBW^*-algebra*, an *alternative W^*-algebra*, a *W^*-algebra*) we mean a non-commutative JB^*-algebra (respectively, a JB^*-algebra, an alternative C^*-algebra, a C^*-algebra) which is a dual Banach space. Thus JBW^*-algebras are precisely those non-commutative JBW^*-algebras which are commutative (cf. Definition 3.3.1), alternative W^*-algebras are precisely those non-commutative JBW^*-algebras which are alternative (cf. Fact 3.3.2), and therefore W^*-algebras are precisely those non-commutative JBW^*-algebras which are associative. Non-commutative JBW^*-algebras (respectively, alternative W^*-algebras) were incidentally introduced in the paragraph immediately before Proposition 4.2.71 (respectively, in Remark 3.5.40).

We note that, by the Banach–Alaoglu and Krein–Milman theorems, the closed unit ball of a non-commutative JBW^*-algebra has extreme points, so that the implication (iv)⇒(i) in Theorem 4.2.36 applies to get the following.

Fact 5.1.7 *Nonzero non-commutative JBW^*-algebras are unital.*

Lemma 5.1.8 *Let A be an algebra over $\mathbb{K}$, and let I be an ideal of A having a unit e. Then e is an idempotent in A and $I = eA$. Moreover, if A is flexible and power-associative, then e is central in A.*

Proof Clearly e is an idempotent in A and we have $eA \subseteq I = eI \subseteq eA$, hence $I = eA$. Suppose that A is flexible and power-associative. Let x be in $A_{\frac{1}{2}}(e)$. Then $e \bullet x = \frac{1}{2}x$, so $e \bullet x = e \bullet (e \bullet x) = \frac{1}{2} e \bullet x$ because $e \bullet x \in I$, and so $x = 0$. Thus $A_{\frac{1}{2}}(e) = 0$. Since $A_{\frac{1}{2}}(e) = (A^{\mathrm{sym}})_{\frac{1}{2}}(e)$, and A^{sym} is power-associative (cf. Corollary 2.4.18), it follows from Lemma 3.1.14 that e is central in A^{sym}. By Corollary 4.3.48, e is central in A. □

§5.1.9 Given a dual Banach space X over $\mathbb{K}$, any w^*-closed subspace M of X will be considered canonically as a new dual Banach space. Indeed, by the bipolar theorem, such a subspace M must be the polar in X of its prepolar $M_\circ$ in X_*, and consequently we have $M = (M_\circ)^\circ \equiv (X_*/M_\circ)'$. With this convention it becomes clear that the weak* topology of M coincides with the restriction to M of the weak* topology of X.

Fact 5.1.10 *Let A be a non-commutative JBW^*-algebra, and let I be a w^*-closed ideal of A. Then:*

(i) *There exists a central idempotent $e \in A$ such that $I = eA$.*
(ii) *I is an M-summand of A.*

Proof By Proposition 3.4.13, I is $*$-invariant, and hence it is a new non-commutative JBW^*-algebra. Therefore, by Fact 5.1.7, I has a unit e, and the proof of assertion (i) is concluded by applying Lemma 5.1.8. Now it turns out that $J := (\mathbf{1}-e)A$ is an ideal of A and that $A = I \oplus J$. Hence I is a direct summand of A, and the proof of assertion (ii) is concluded by invoking Lemma 5.1.4. □

The proof we have just given shows that, as happens with the unit of any $*$-algebra, the idempotent e must be self-adjoint. As we note in the next remark, this is not new for us.

Remark 5.1.11 Let A be a non-commutative JB^*-algebra. We already know that central idempotents of A are self-adjoint (cf. Fact 3.3.4, §4.3.38, and either Theorem 4.3.47 or Corollary 4.3.48). Nevertheless, the most natural verification of this result consists of noticing that the centre $Z(A)$ of A is a commutative C^*-algebra (cf. Proposition 3.4.1(i)), and of applying then to $Z(A)$ the commutative Gelfand–Naimark theorem. Another proof, close to that of Fact 5.1.10, is the following.

> Let e be a central idempotent of A. Then $I := eA$ is an ideal of A, and is closed in A because $I = \{a \in A : a = ea\}$. Therefore, by Proposition 3.4.13, I is $*$-invariant. Since e is a unit for I, it follows that $e^* = e$.

The notions of L-summand and of M-ideal of a normed space were incidentally introduced in Subsection 2.9.4. Now we are going to recall and develop them in a more detailed way.

Definition 5.1.12 Let X be a normed space over $\mathbb{K}$.

(i) By an *L-projection* on X we mean a linear projection $P : X \to X$ such that

$$\|x\| = \|P(x)\| + \|x - P(x)\| \text{ for every } x \in X.$$

(ii) A subspace Y of X is said to be an *L-summand* of X if Y is the range of an L-projection on X.

(iii) By an M-ideal of X we mean a closed subspace Y of X such that Y° is an L-summand of X'.

Some comments on Definitions 5.1.3 and 5.1.12 are in order.

§5.1.13 Let X be a normed space over $\mathbb{K}$. There is an obvious duality between L- and M-projections. Indeed,

$$P \text{ is an } \left\{ \begin{array}{c} L\text{-projection} \\ M\text{-projection} \end{array} \right\} \text{ on } X \text{ if and only if } P' \text{ is an } \left\{ \begin{array}{c} M\text{-projection} \\ L\text{-projection} \end{array} \right\} \text{ on } X'.$$

As a consequence, *M-summands of X are M-ideals of X.*

Proposition 5.1.14 *Let X be a normed space over $\mathbb{K}$. Then any two L- (respectively, M-)projections on X commute.*

Proof Let P and Q be L-projections on X. Then for $x \in X$ we have

$$\begin{aligned}\|Q(x)\| &= \|PQ(x)\| + \|(I_X - P)Q(x)\| \\ &= \|QPQ(x)\| + \|(I_X - Q)PQ(x)\| + \|Q(I_X - P)Q(x)\| \\ &\quad + \|(I_X - Q)(I_X - P)Q(x)\| \\ &= \|QPQ(x)\| + \|Q(x) - QPQ(x)\| + 2\|PQ(x) - QPQ(x)\| \\ &\geq \|Q(x)\| + 2\|PQ(x) - QPQ(x)\|,\end{aligned}$$

so that $PQ = QPQ$. But likewise we obtain $P(I_X - Q) = (I_X - Q)P(I_X - Q)$ which is equivalent to $QP = QPQ$. Therefore $PQ = QP$.

Now that we know that any two L-projections commute, the fact that any two M-projections commute follows by invoking §5.1.13. □

Corollary 5.1.15 *Let X be a normed space over $\mathbb{K}$, and let Y be an L- (respectively, M-)summand of X. Then there is a unique L- (respectively, M-)projection on X whose range is Y.*

Lemma 5.1.16 *Let X be a nonzero complex Banach space, and let P be an L- or M-projection on X. Then $P \in H(BL(X), I_X)$.*

Proof In both cases there is a function $f : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\|y + z\| = f(\|y\|, \|z\|) \text{ for all } y \in P(X) \text{ and } z \in \ker(P).$$

Let r be in $\mathbb{R}$. Then $\exp(irP) = e^{ir}P + I_X - P$. Therefore for every $x \in X$ we have

$$\|\exp(irP)(x)\| = \|e^{ir}P(x) + (I_X - P)(x)\| = f(\|P(x)\|, \|(I_X - P)(x)\|) = \|x\|.$$

Thus $\|\exp(irP)\| = 1$, and Corollary 2.1.9(iii) concludes the proof. □

Proposition 5.1.17 *Let X be a nonzero complex Banach space, let Y be an M-ideal of X, and let T be a hermitian operator on X. Then $T(Y) \subseteq Y$.*

Proof Let P be the L-projection onto the polar Y° of Y in X'. Then the transpose operator T' of T is also a hermitian operator on X' (cf. Corollary 2.1.3) so, for each $t \in \mathbb{R}$, $\exp(itT')$ is a surjective linear isometry on X' (cf. Corollary 2.1.9(iii)). It follows easily that $\exp(itT')P\exp(-itT')$ is a new L-projection on X'. By Proposition 5.1.14, we realize that $[\exp(itT')P\exp(-itT'), P] = 0$. Computing the coefficient of t in the power-series development of the left-hand side, we have $[[T', P], P] = 0$. Since L-projections are hermitian operators (cf. Lemma 5.1.16), it follows from Corollary 2.4.3 that $[T', P] = 0$. Which implies that $T'(Y^\circ) \subseteq Y^\circ$, and finally $T(Y) \subseteq Y$. □

Fact 5.1.18 *Let E and F be topological spaces, let x be in E, and let $f : E \to F$ be a function such that $f(x)$ is a cluster point of $f(x_\lambda)$ whenever x_λ is any net in E converging to x. Then f is continuous at x.*

Proof Assume that f is not continuous at x. Let Λ stand for the set of all neighbourhoods of x in E ordered by reverse inclusion. Then there exists a neighbourhood

N of $f(x)$ in F such that for every $\lambda \in \Lambda$ we can find $x_\lambda \in \lambda$ with $f(x_\lambda) \notin N$. It becomes clear that $\lim_\lambda x_\lambda = x$ and that $f(x)$ is not a cluster point of the net $f(x_\lambda)$, contrary to the assumption. □

A celebrated theorem of S. Banach (sometimes attributed to M. Krein and V. Šmulyan) asserts that *a linear form on the dual* X' *of a Banach space* X *is* w^**-continuous if* (and only if) *so is its restriction to* $\mathbb{B}_{X'}$ (see for example [1161, Corollary 3.11.4]). An apparently more general formulation of this result is collected in the following.

Fact 5.1.19 *Let* X *be a Banach space over* $\mathbb{K}$*, let* Y *be a normed space over* $\mathbb{K}$*, and let* $T : X' \to Y'$ *be a linear or conjugate-linear mapping whose restriction to* $\mathbb{B}_{X'}$ *is* w^**-continuous. Then* T *is* w^**-continuous.*

Proof It is enough to show that for each $y \in Y$ the linear form f on X' defined by $f(x') := T(x')(y)$ (or $f(x') := \overline{T(x')(y)}$) is w^*-continuous. But this follows from the assumption that $T_{|\mathbb{B}_{X'}}$ is w^*-continuous and the Banach theorem quoted immediately above. □

Corollary 5.1.20 *Let* X *be a Banach space over* $\mathbb{K}$*, and let* P *be a linear projection on* X'*. Then* P *is* w^**-continuous if* (and only if) P *is bounded and both* $P(X')$ *and* $\ker(P)$ *are* w^**-closed in* X'*.*

Proof Suppose that P is bounded and that $P(X')$ and $\ker(P)$ are w^*-closed in X'. Let x' be in $\mathbb{B}_{X'}$, and let x'_λ be a net in $\mathbb{B}_{X'}$ w^*-convergent to x'. Take a cluster point y' of the net $P(x'_\lambda)$ in the weak* topology. Then $x' - y'$ is a cluster point of the net $x'_\lambda - P(x'_\lambda)$ in the weak* topology. Since $P(X')$ and $\ker(P)$ are w^*-closed in X', it follows that $y' \in P(X')$ and $x' - y' \in \ker(P)$. Therefore $y' = P(x')$, and hence $P(x')$ is a cluster point of the net $P(x'_\lambda)$ in the weak* topology. Keeping in mind the arbitrariness of $x' \in \mathbb{B}_{X'}$ and of the net x'_λ w^*-convergent to x', it follows from Fact 5.1.18 that $P_{|\mathbb{B}_{X'}}$ is w^*-continuous. Finally, by Fact 5.1.19, P is w^*-continuous. □

The reader might wonder why we did not introduce the notion of an 'L-ideal' of a normed space X over $\mathbb{K}$, meaning a closed subspace of X whose polar is an M-summand of X'. The reason is given by the following.

Lemma 5.1.21 *Let* X *be a Banach space over* $\mathbb{K}$*. Then we have:*

(i) *M-summands of X' are w^*-closed.*
(ii) *M-projections on X' are w^*-continuous.*
(iii) *L-summands of X are precisely those closed subspaces of X whose polars are M-summands of X'.*

Proof To prove assertion (i), let us consider a decomposition $X' = Y \oplus_\infty Z$, and let us assume to the contrary that Y is not w^*-closed. In this case, by the Krein–Šmulyan theorem (see for example [778, Corollary 2.7.12]), there exists a net x'_λ in $\mathbb{B}_Y$ w^*-convergent to some $x' \in Z$, $x' \neq 0$. Then $y'_\lambda := x'_\lambda + \frac{x'}{\|x'\|}$ defines a net in $\mathbb{B}_{X'}$ whose w^*-limit has norm $1 + \|x'\| > 1$, a contradiction.

Assertion (ii) follows from assertion (i) and Corollary 5.1.20.
Assertion (iii) follows from assertion (ii) and §5.1.13. □

Theorem 5.1.22 *Let A be a non-commutative JB^*-algebra. Then:*

(i) *The M-ideals of A are precisely the closed ideals of A.*
(ii) *The M-summands of A are precisely the direct summands of A.*

Proof Let I be an M-ideal of A. For $a \in A$, let T_a stand for either L_a or R_a. Let x be in $H(A,*)$. Then, by Lemma 3.6.24 and Proposition 5.1.17, we have $T_x(I) \subseteq I$, i.e. $xI, Ix \subseteq I$. Since every element $a \in A$ can be written as $a = x + iy$ with $x, y \in H(A,*)$, it follows that I is an ideal of A. Now let I be a closed ideal of A. We know that A'' is a unital non-commutative JBW^*-algebra with separately w^*-continuous product (cf. Theorem 3.5.34), and that consequently $I^{\circ\circ}$ is an ideal of A''. Since $I^{\circ\circ}$ is w^*-closed, it follows from Fact 5.1.10 that $I^{\circ\circ}$ is an M-summand of A''. Therefore, by Lemma 5.1.21(iii), I is an M-ideal of A. This concludes the proof of assertion (i).

We already proved in Lemma 5.1.4 that direct summands of A are M-summands. Conversely, let I be an M-summand of A. Let P be the M-projection onto I. Then both I and $J := (I_A - P)(A)$ are M-ideals of A with $A = I \oplus J$. Therefore, by assertion (i), I and J are ideals of A. Thus I is a direct summand of A. This concludes the proof of assertion (ii). □

§5.1.23 Let X be a dual Banach space over $\mathbb{K}$. We say that X_* is the unique predual of X if, whenever Z is any Banach space over $\mathbb{K}$ such that $Z' = X$, and we see X_* and Z as subspaces of X' via the corresponding canonical embeddings, we have $X_* = Z$. Formulating this notion in a slightly more precise way, as is done in [954, Définition 1], the following fact becomes a tautology.

Fact 5.1.24 *Let X and Y be dual Banach spaces over $\mathbb{K}$ having a unique predual. Then surjective linear isometries from X to Y are w^*-continuous.*

Now we recall the definition of the order in a JB-algebra, together with some related results.

§5.1.25 Let A be a JB-algebra. Since the unital extension $A_{\mathbb{1}}$ of A becomes a unital JB-algebra with $n(A_{\mathbb{1}}, \mathbb{1}) = 1$ (cf. Corollary 3.1.11 and Proposition 3.1.4(iii)), $\mathbb{1}$ is a vertex of $\mathbb{B}_{A_{\mathbb{1}}}$, and hence A enjoys of the order induced by the numerical-range order of $(A_{\mathbb{1}}, \mathbb{1})$ as defined in §2.3.34. Indeed, an element a in A is called positive whenever $V(A_{\mathbb{1}}, \mathbb{1}, a) \subseteq \mathbb{R}_0^+$. The set A^+ of all positive elements in A is a closed proper convex cone in A, and the order in A is given by:

$$a \leq b \;\text{ if and only if }\; b - a \in A^+.$$

In the case that A is unital, the passing to the unital extension is unnecessary, i.e. the order in A, as defined above, coincides with the numerical-range order of $(A, \mathbf{1})$ (cf. §3.1.27). Anyway, if a, b are in A, and if $0 \leq a \leq b$, then $\|a\| \leq \|b\|$ (cf. Fact 2.3.36). Moreover, according to Lemma 3.1.29, we have $A^+ = \{a^2 : a \in A\}$ (which puts in agreement our definition of the order with the one given in [738, 3.3.3]), and

$$U_a(A^+) \subseteq A^+ \;\text{ for every }\; a \in A. \tag{5.1.1}$$

Definition 5.1.26 Let A be a JB-algebra. A linear functional f on A is said to be *positive* if $f(a) \geq 0$ whenever a is a positive element of A. The JB-algebra A is said to be *monotone complete* if each bounded increasing net a_λ in A has a least upper bound a in A. Suppose that A is monotone complete. A linear functional f on A is called *normal* if it is bounded and if $f(a_\lambda) \to f(a)$ for each net a_λ as above.

Now we involve in our development the following outstanding result whose proof is omitted.

Theorem 5.1.27 [738, Theorem 4.4.16] *Let A be a JB-algebra. Then the following conditions are equivalent:*

(i) *A is monotone complete and the set of all positive normal linear functionals on A separates the points of A.*
(ii) *A is a dual Banach space.*

Moreover, if the above conditions are fulfilled, then the predual of A is unique and consists of the normal linear functionals on A.

According to the above theorem and the definition of a JBW-algebra in [738, 4.1.1] (as those JB-algebras A satisfying condition (i) above), we introduced JBW-algebras as those JB-algebras which are dual Banach spaces (cf. the paragraph immediately before Proposition 3.1.12).

§5.1.28 Let A be a non-commutative JB^*-algebra. Then $H(A,*)$ becomes a JB-algebra in a natural way (cf. Corollary 3.4.3), and hence, as we agreed in §3.4.68, it will be seen endowed with the order remembered in §5.1.25. By the sake of shortness, the order of $H(A,*)$ is called the order of A, positive elements of $H(A,*)$ are called positive elements of A, and we set $A^+ := H(A,*)^+$. Thus, an element $a \in A$ is positive if and only if $a = h^2$ for some $h \in H(A,*)$. As a consequence, we have that

$$a^* \bullet a \geq 0 \text{ for every } a \in A. \tag{5.1.2}$$

We note that, by Fact 4.1.67(ii) and Proposition 4.5.17(ii),

$$A^+ = \{h \in H(A,*) : \text{J-sp}(A_{\mathbb{1}}, h) \subseteq \mathbb{R}_0^+\},$$

and that, if A is unital, then we have in fact

$$A^+ = \{h \in H(A,*) : \text{J-sp}(A, h) \subseteq \mathbb{R}_0^+\}.$$

Needless to say that, in the case that A is actually a C^*-algebra, we find back the usual order on A (cf. §1.2.47 and Proposition 2.3.39(i)).

Theorem 5.1.29 *Let A be a non-commutative JBW^*-algebra. Then we have:*

(i) *$H(A,*)$ is w^*-closed in A, and hence is a JBW-algebra in a natural way* (cf. Corollary 3.4.3 and §5.1.9).
(ii) *The involution of A is w^*-continuous.*

(iii) $A_* = H(A,*)_* \oplus iH(A,*)_*$, *meaning that, for each* $h_* \in H(A,*)_*$, *the mapping*

$$h + ik \to h_*(h) + ih_*(k) \quad (h,k \in H(A,*))$$

belongs to A_*, *and that, for each* $a_* \in A_*$, *there exist unique functionals* $h_*, k_* \in H(A,*)_*$ *such that*

$$a_*(h+ik) = h_*(h) - k_*(k) + i(h_*(k) + k_*(h)) \ \textit{for all} \ \ h,k \in H(A,*).$$

(iv) *A has a unique predual.*
(v) *The positive part* A^+ *of A is* w^**-closed in A.*
(vi) *A equals the norm-closed linear hull of the set of its self-adjoint idempotents.*

Proof We may suppose that $A \neq 0$. Then, by Fact 5.1.7, A is unital.

In view of the Krein–Šmulyan theorem, to prove assertion (i) it is enough to show that $H(A,*) \cap \mathbb{B}_A$ is w^*-closed in A. We argue by contradiction, so that there exists a net h_λ in $H(A,*) \cap \mathbb{B}_A$ w^*-convergent to $h+ik$ with $h,k \in H(A,*)$ and $k \neq 0$. Replacing h_λ with $-h_\lambda$ if necessary, we may suppose that there is a positive number α in J-sp(A,k) (cf. Fact 4.1.67(i) and Corollary 4.1.72(i)). Take $n \in \mathbb{N}$ such that

$$n > \frac{1-\alpha^2}{2\alpha}. \tag{5.1.3}$$

Noticing that, for each λ, the closed subalgebra of A generated by h_λ and $\mathbf{1}$ is a C^*-algebra (cf. Proposition 3.4.1(ii)), we have

$$\|h_\lambda + in\mathbf{1}\| = \|(h_\lambda + in\mathbf{1})^*(h_\lambda + in\mathbf{1})\|^{\frac{1}{2}} = \|h_\lambda^2 + n^2\mathbf{1}\|^{\frac{1}{2}} \leq (\|h_\lambda^2\| + n^2)^{\frac{1}{2}} \leq (1+n^2)^{\frac{1}{2}}.$$

Therefore, since $h + ik = w^*\text{-}\lim h_\lambda$, we conclude that

$$\|h + ik + in\mathbf{1}\| \leq (1+n^2)^{\frac{1}{2}}. \tag{5.1.4}$$

On the other hand, since $H(A,*)$ is a JB-algebra in a natural way (cf. Corollary 3.4.3), and the involution of A is an isometry (cf. Proposition 3.3.13), we have

$$\begin{aligned}\|k+n\mathbf{1}\| &= \|(k+n\mathbf{1})^2\|^{\frac{1}{2}} \leq \|(k+n\mathbf{1})^2 + h^2\|^{\frac{1}{2}} \\ &= \|(h+i(k+n\mathbf{1}))^* \bullet (h+i(k+n\mathbf{1}))\|^{\frac{1}{2}} \leq \|h+i(k+n\mathbf{1})\|. \end{aligned} \tag{5.1.5}$$

Finally, since $\alpha + n \leq \|k+n\mathbf{1}\|$ (because $\alpha \in$ J-sp(A,k) and Theorem 4.1.17 applies), it is enough to invoke (5.1.5), (5.1.4), and (5.1.3) to get

$$\|k+n\mathbf{1}\| \leq \|h+i(k+n\mathbf{1})\| \leq (1+n^2)^{\frac{1}{2}} < \alpha + n \leq \|k+n\mathbf{1}\|,$$

the desired contradiction.

Keeping in mind that $P := \frac{1}{2}(I_A + *)$ is a bounded linear projection on the real Banach space underlying that of A satisfying $P(A) = H(A,*)$ and $\ker(P) = iH(A,*)$, assertion (ii) follows from assertion (i) and Corollary 5.1.20.