NON-ASSOCIATIVE NORMED ALGEBRAS

Volume 1: The Vidav–Palmer and Gelfand–Naimark Theorems

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Encyclopedia of Mathematics and its Applications

Non-Associative Normed Algebras

Volume 1: The Vidav–Palmer and Gelfand–Naimark Theorems

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Contents

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Miguel Cabrera García and Ángel Rodríguez Palacios
Frontmatter
More information

Preface

Reviewing the non-associative part of a monograph by Irving Kaplansky

In 1970, Irving Kaplansky published his small monograph [762] on *Algebraic and analytic aspects of operator algebras*, and devoted its last section to providing the reader with his impressions concerning non-associative normed algebras. Actually, he began the section by saying:

> I predict that when the time is ripe there is going to be quite a flurry of activity concerning nonassociative Banach algebras in general, and nonassociative C^*-algebras in particular. Let me take the space to speculate a little on what we may see some day.

Many years have passed since the publication of [762] and, as a matter of fact, most of Kaplansky's predictions have come true, some even exceeding the original expectations. The diverse results corroborating the accuracy of Kaplansky's predictions will be used to illustrate the content of the book we are introducing. Therefore, let us continue reproducing Kaplansky's words in short excerpts, and insert some clarifying comments.

> The speculation can start encouragingly, with a fact. The (complex) Gelfand–Mazur Theorem works fine: a normed division algebra must be the complex numbers. Of course, we must agree on what a division algebra A is to be. We take it to mean that for any nonzero x, both R_x and L_x are one-to-one and onto, where R_x (L_x) denotes right (left) multiplication by x. (Actually, for the proof all we need is R_x.)

With the words 'for the proof all we need is R_x', Kaplansky is suggesting the notion of a right-division algebra.

> Suppose then that A is a normed division algebra. We claim that A is one-dimensional, and by way of contradiction we assume that x and y are linearly independent. Then for every complex scalar λ, $R_{x-\lambda y} = R_x - \lambda R_y$ is a bounded operator on A which is one-to-one and onto. The same is true of $R_x R_y^{-1} - \lambda I$. But $R_x R_y^{-1}$ must have something in its spectrum.

It seems obvious to us that Kaplansky implicitly assumes that the (possibly non-associative) normed algebra A above is complete, to be sure that 'the same is true of $R_x R_y^{-1} - \lambda I$' concerning boundedness. Under this assumption, Kaplansky's claim (that complete normed one-sided division complex algebras are isomorphic to $\mathbb{C}$) is stated in Corollary 2.7.3, whereas the non-complete case is raised as Problem 2.7.4 (see also Theorem 4.1.63 for a partial affirmative answer). Actually, a result better than Kaplansky's claim holds. Indeed, if A is a complete normed complex algebra, and if it is a quasi-division algebra (which means that, for every nonzero $x \in A$, at least one of the operators L_x, R_x is one-to-one and onto), then $\dim(A) \leqslant 2$ (Theorem 2.7.7) and, in general, no more can be said (Example 2.5.36). Moreover, if in addition A is unital or nearly associative, then A is isomorphic to $\mathbb{C}$ (Corollaries 2.7.9 and 2.7.10). It is also worth mentioning that, for associative (and even alternative) algebras, the notions of division, one-sided division, and quasi-division coincide, and also coincide with the classical notion of a division algebra in this setting – namely that the algebra is unital and each nonzero element has an inverse (Proposition 2.5.38).

Let us continue with Kaplansky's words:

> On the other hand, the real Gelfand–Mazur theorem does not seem to have received any attention. The conjecture is that any real normed division algebra is finite-dimensional, after which the topologists would teach us that the dimension is 1, 2, 4, or 8.

The conjecture that normed division real algebras are finite-dimensional was first formulated by Wright [640], after proving it in the particular case of absolute-valued algebras (Corollary 2.6.24). The general case of the conjecture, even with the additional requirement of completeness, remains open (Problem 2.7.45). Nevertheless, normed one-sided division real algebras need not be finite-dimensional, even if they are absolute-valued and complete (Theorem 2.7.38). The enormous theorem of 'the topologists' (that finite-dimensional division real algebras are of dimension 1, 2, 4, or 8) is stated without proof in Theorem 2.6.51, referring the reader to the whole of Chapter 11 of [727] for a complete proof. The particularization to absolute-valued algebras is much more elementary, and is stated in Fact 2.6.50.

Kaplansky continues as follows:

> There is a related circle of ideas which has received a good deal of attention. In [337], Inglestam proved the following pretty theorem: if an associative real Banach algebra A is built on a Hilbert space and has a unit of norm 1, then in the first place A is finite-dimensional; moreover A must be the reals, complexes, or quaternions in their ordinary norm. An alternative account was given by Smiley [590]. The crucial point here is the behaviour of the unit element relative to convexity, and nonassociative generalizations have been given by Strzelecki [605, 606] and Inglestam [338]. (One should note the difference between this problem and that of Urbanik and Wright [620], who do not assume a Hilbert space but deduce it from the equality $\|xy\| = \|x\|\,\|y\|$.)

Ingelstam's theorem (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras whose norm comes from an inner product) is stated in Corollary 2.6.22. Here $\mathbb{H}$ and $\mathbb{O}$ stand for the algebra of Hamilton's quaternions and the algebra of Cayley numbers, respectively. Strzelecki's generalization (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras whose closed unit ball has a unique tangent hyperplane at the unit) is stated as the equivalence (ii)$\Leftrightarrow$(iii) in Theorem 2.6.21.

In our opinion, the non-commutative Urbanik–Wright theorem (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique unital absolute-valued real algebras) becomes one of the jewels of the theory of non-associative normed algebras. Therefore, we devote special attention to it. We state it as the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.6.21, and provide a second proof in §2.7.67. The non-commutative Urbanik–Wright theorem had also been predicted by Kaplansky in [377], who, by means of a general theorem on the so-called unital composition algebras, reduced the proof to the case that the norm comes from an inner product. The two proofs we have given of the non-commutative Urbanik–Wright theorem follow Kaplansky's indication. We also state the commutative Urbanik–Wright theorem, proved in [620] as well, that $\mathbb{R}$, $\mathbb{C}$, and another natural two-dimensional algebra, are the unique absolute-valued commutative real algebras (Theorem 2.6.41).

> Now let us go beyond Gelfand–Mazur considerations to general theory. I divide the remarks under three headings, corresponding to the three classes of nonassociative algebras that have withstood the test of time.
>
> (1) *Lie algebras*. Let me be honest. I have nothing to say, even by way of the wildest speculation, about a possible theory of Banach Lie algebras.

Banach Lie algebras will be discussed in our book only in an incidental way. Nevertheless, let us say that a complete structure theory for Lie H^*-algebras has been developed (cf. Remark 2.6.54 for references), and that different aspects of general or particular Banach Lie algebras have been considered in [47, 96, 97, 98, 99, 130, 175, 188, 253, 254, 452, 453, 569, 604, 615, 627]. For a comprehensive account, the reader is referred to the books [687, 688, 740].

> (2) *Alternative*. Alternative rings are only a slight generalization of associative rings. It is therefore a reasonable presumption that most standard associative results will survive, perhaps in a suitably altered form, and perhaps with a lot of extra proof.

The 'reasonable presumption' above is indeed correct. As a relevant sample, Kaplansky's celebrated theorem [375], that normed associative real algebras with no nonzero topological divisor of zero are isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, has its 'suitably altered form' for alternative algebras (Theorem 2.5.50).

> Alternative C^*-algebras look like a plausible topic. Over the complex numbers the essentially new algebra – the Cayley matrix algebra – has every right to be called a C^*-algebra. But the subject should be developed in real style, so as to allow the Cayley division algebra to survive.

We feel that Kaplansky assumes the current Gelfand–Naimark characterization of closed $*$-invariant subalgebras of operators on complex Hilbert spaces and that consequently, by an 'alternative C^*-algebra', he means a complete normed alternative complex algebra A endowed with a conjugate-linear algebra involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$. If this is so, then the complex Cayley matrix algebra $C(\mathbb{C})$ is indeed an alternative C^*-algebra (Proposition 2.6.8). Moreover, $C(\mathbb{C})$ becomes the unique 'essentially new' alternative C^*-algebra. Indeed, $C(\mathbb{C})$ is the unique prime alternative C^*-algebra which is not associative and, by a standard C^*-argument, the theory of general alternative C^*-algebras reduces to the cases of prime associative C^*-algebras and that of $C(\mathbb{C})$ [125, 481]. It is also worth mentioning that, in the case of unital algebras, there are relevant redundancies in the definition of an alternative C^*-algebra suggested above. The redundances are so severe that, as we prove in Theorem 3.2.5, the alternative identities $x^2y = x(xy)$ and $yx^2 = (yx)x$ follow from the remaining requirements (see also Theorem 3.5.53 for a non-unital variant).

As far as we know, a systematic treatment of real alternative C^*-algebras has not been done to date. Anyway, real alternative C^*-algebras can be introduced as closed $*$-invariant real subalgebras of complex ones (see Definition 4.2.45). With this convention, clearly, the algebra $\mathbb{O}$ of Cayley numbers 'survives'.

> Alternative AW^*-algebras probably have a decisive structure theory. (I feel sure that subject will be developed in AW^* rather than in W^* style – unless somebody cares to work up the theory of Cayley Hilbert space.) There ought to be a unique direct sum decomposition into four homogeneous pieces: (a) real Cayley matrix, (b) Cayley division, (c) complex Cayley matrix, (d) associative.

As a matter of fact, concerning 'AW^* style', no progress has been made on the above prediction, even in the complex case. However, the 'W^* style' has become extremely successful. Indeed, as shown by G. Horn [331], every (complex) alternative W^*-algebra has a unique direct sum decomposition into two pieces: (a) the algebra of all continuous functions from a suitable compact Hausdorff hyper-Stonean space to $C(\mathbb{C})$; (b) an associative von Neumann algebra. It can be derived from Horn's theorem that every real alternative W^*-algebra has 'a unique direct sum decomposition into four pieces': (a) the algebra of all continuous functions from a suitable compact Hausdorff hyper-Stonean space to the real Cayley matrix algebra $C(\mathbb{R})$; (b) the same with $\mathbb{O}$ instead of $C(\mathbb{R})$; (c) the same with $C(\mathbb{C})$ instead of $C(\mathbb{R})$; (d) a real associative von Neumann algebra. Thus, 'in W^* style', Kaplansky's prediction is right.

> (3) *Jordan*. In saying that Jordan Banach algebras ought to be studied we have the blessing of the Master himself [461]. In recent years Topping [813, 614], Effros and Størmer [228] and Størmer [601, 602, 603] have made significant progress. In one way, however, they made an undesirable retreat from [461]. By assuming from the start that they were dealing with operators on a Hilbert space, they ruled out the exceptional Jordan algebra, and left to the future the possibility of a theorem asserting that suitable infinite-dimensional Jordan Banach algebras are special. Admittedly I am prejudiced, but perhaps the AW^* point of view is a good one here.

Eight years after the above paragraph was written, Alfsen, Shultz, and Størmer [15] removed the 'undesirable retreat from [461]' by introducing the so-called JB-algebras. JB-algebras are complete normed Jordan real algebras which include both Jordan algebras of self-adjoint 'operators on a Hilbert space' (Corollary 3.1.2) and 'the exceptional Jordan algebra' (Example 3.1.56). JB-algebras enjoy a deep and complete structure theory which has been comprehensively developed in Hanche-Olsen and Størmer [738], and has been revisited recently in Alfsen and Shultz [673]. Different approaches to JB-algebras can be found in Ayupov, Rakhimov, and Usmanov [684], and Iochum [748]. Since we are unable to organize the basic theory of JB-algebras in a better way than that of [738], we have limited ourselves to state without proof those results which are needed for our actual purposes, and to complement the theory in some aspects (originated by the papers of Wright [641] and Wright and Youngson [643]) which are not covered by the Hanche-Olsen and Størmer book. This is done from Section 3.1. Among the results taken from [738], we emphasize the one asserting that JB-algebras generated by two elements are (isometrically isomorphic to) Jordan algebras of self-adjoint operators on a

Hilbert space (Proposition 3.1.3). Thus, 'suitable infinite-dimensional Jordan Banach algebras are special'.

Kaplansky concludes with the following comment:

> Kadison's beautiful results [358, 359] on isometries of C^*-algebras deserve to be generalized to Jordan C^*-algebras if only to encompass the exceptional Jordan algebra.

What Kaplansky means by a 'Jordan C^*-algebra' is not in doubt because he himself later introduced this notion in detail in his final lecture to the 1976 St. Andrews Colloquium of the Edinburgh Mathematical Society, and pointed out its potential importance. Jordan C^*-algebras (called JB^*-algebras since Youngson's paper [652]) were first studied by Wright [641], who proved that the passing from each JB^*-algebra to its self-adjoint part establishes a bijective categorical correspondence between JB^*-algebras and JB-algebras (see Corollary 3.4.3 and Theorem 3.4.8).

Kadison's celebrated Theorem A of [358] is fully discussed in Section 2.2. Actually, we prove the unit-free Paterson–Sinclair generalization [480] of Kadison's result (Theorem 2.2.19), starting from a non-associative germ (Theorem 2.2.9). An appropriate version for JB-algebras of the Kadison–Paterson–Sinclair theorem, following the arguments in [643, 223, 342], is stated in Theorem 3.1.21. Therefore, as first proved in [643], for isometries preserving units of unital JB^*-algebras, Kadison's theorem survives (see Proposition 3.4.25) 'encompassing the exceptional Jordan algebra'. Nevertheless, Kadison's theorem does not survive for general isometries, nor even in the case of closed $*$-invariant unital Jordan subalgebras of operators on Hilbert spaces. The reason is that, as shown by Braun, Kaup, and Upmeier [126], two such algebras can be linearly isometric without being $*$-isomorphic (see Antitheorem 3.4.34). Anyway, the Kadison–Paterson–Sinclair theorem survives verbatim in the case of alternative C^*-algebras and also, in a suitably altered form, in the case of JB^*-algebras which are dual Banach spaces [366].

About the core of the book

Now that we have concluded our review of the non-associative part of Kaplansky's monograph [762], let us comment about the leitmotiv of the present book. (To this end, we found the Introduction of [366] useful.) Our aim is to deal with non-associative generalizations of C^*-algebras. To this end, we realize that most generalizations appearing in the literature, like JB^*-algebras, JB-algebras (both had already been discussed when we reviewed [762]), and JB^*-triples, contain C^*-algebras, but only after suitable manipulations. Thus C^*-algebras become JB^*-algebras after replacing the associative product xy with the Jordan product

$$x \bullet y := \frac{1}{2}(xy + yx).$$

They are JB-algebras after the same replacement and then passing to the self-adjoint part, and they are also JB^*-triples after replacing the product with the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

As a matter of fact, many years ago we tried to approach the non-associative generalizations of C^*-algebras in a somewhat more ingenuous way. Indeed, we removed associativity in the abstract characterizations of unital (associative) C^*-algebras given either by the Gelfand–Naimark theorem or by the Vidav–Palmer theorem, and studied (possibly non-unital) closed $*$-invariant subalgebras of the Gelfand–Naimark or Vidav–Palmer algebras born after removing associativity.

To be more precise, for a norm-unital complete normed (possibly non-associative) complex algebra A, we considered the following conditions:

(GN) (Gelfand–Naimark axiom). *There is a conjugate-linear vector space involution $*$ on A satisfying $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a\|^2$ for every a in A.*

(VP) (Vidav–Palmer axiom). $A = H(A,\mathbf{1}) + iH(A,\mathbf{1})$.

In both conditions, $\mathbf{1}$ denotes the unit of A, whereas, in (VP), $H(A,\mathbf{1})$ stands for the closed real subspace of A consisting of those elements h in A such that $f(h)$ belongs to $\mathbb{R}$ for every bounded linear functional f on A satisfying $\|f\| = f(\mathbf{1}) = 1$.

As we said before, if the norm-unital complete normed complex algebra A above is associative, then (GN) and (VP) are equivalent conditions, both providing nice characterizations of unital C^*-algebras (see Lemma 2.2.5 and Theorems 1.2.3 and 2.3.32). In the general non-associative case we were considering, things began to be more amusing. Indeed, it is easily seen that (GN) implies (VP) (refer again to Lemma 2.2.5), but the converse implication is not true (see Example 2.3.65).

The amusing aspect of the non-associative consideration of the Vidav–Palmer and the Gelfand–Naimark axioms greatly increased thanks to the fact (explained in what follows) that Condition (VP) (respectively, (GN)) on a norm-unital complete normed complex algebra A implies that A is ‘nearly’ (respectively, ‘very nearly’) associative. To specify our last assertion, let us recall some elementary concepts of non-associative algebra. *Alternative algebras* are defined as those algebras A satisfying $a^2b = a(ab)$ and $ba^2 = (ba)a$ for all a,b in A. By Artin’s theorem (stated in Theorem 2.3.61), an algebra A is alternative (if and) only if, for all a,b in A, the subalgebra of A generated by $\{a,b\}$ is associative. According to Definition 2.4.9 and Proposition 3.2.1, *non-commutative Jordan algebras* can be introduced as those algebras A satisfying the *Jordan identity* $(ab)a^2 = a(ba^2)$ and the *flexibility* condition $(ab)a = a(ba)$. As shown in Proposition 2.4.19, non-commutative Jordan algebras are *power-associative* (i.e. all subalgebras generated by a single element are associative) and, as a consequence of Artin’s theorem, alternative algebras are non-commutative Jordan algebras. For an element a in an algebra A, we denote by U_a the mapping $b \to a(ab+ba) - a^2b$ from A to A. By means of Definitions I and II below, we provide the algebraic notions just introduced with analytic robes. We note that the notion of an alternative C^*-algebra, emphasized in Definition I, had already appeared when we reviewed the monograph [762].

Definition I By an *alternative C^*-algebra* we mean a complete normed alternative complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every a in A.

Definition II By a *non-commutative JB^*-algebra* we mean a complete normed non-commutative Jordan complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_a(a^*)\| = \|a\|^3$ for every a in A.

Since the equality $U_a(b) = aba$ holds for all elements a,b in an alternative algebra, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. Actually, alternative C^*-algebras are precisely those non-commutative JB^*-algebras which are alternative (see Fact 3.3.2). Now, by means of Theorems GN and VP which follow, we can specify how the behaviour of the Gelfand–Naimark and Vidav–Palmer axioms in the non-associative setting were clarified.

Theorem GN *Norm-unital complete normed complex algebras fulfilling the Gelfand–Naimark axiom are nothing other than unital alternative C^*-algebras.*

Theorem VP *Norm-unital complete normed complex algebras fulfilling the Vidav–Palmer axiom are nothing other than unital non-commutative JB^*-algebras.*

Now, keeping in mind Theorems GN and VP above, together with the obvious fact that closed $*$-invariant subalgebras of an alternative C^*-algebra (respectively, of a non-commutative JB^*-algebra) are alternative C^*-algebras (respectively, non-commutative JB^*-algebras), there is no doubt that, even in the non-unital case, both alternative C^*-algebras and non-commutative JB^*-algebras become reasonable non-associative generalizations (the latter containing the former) of classical C^*-algebras. Therefore the main goal of our book will be to prove Theorems GN and VP (see Theorems 3.2.5 and 3.3.11), together with their non-unital variants (see Theorem 3.5.53 and [365]), and to describe alternative C^*-algebras and non-commutative JB^*-algebras by means of the so-called representation theory.

It is worth mentioning that although our approach to the non-associative generalizations of C^*-algebras is different from those of JB^*-algebras, JB-algebras, and JB^*-triples, in the end all approaches give rise essentially to the same mathematical creature. Indeed, Kaplansky's JB^*-algebras are nothing other than those non-commutative JB^*-algebras which are commutative. On the other hand, every non-commutative JB^*-algebra becomes a JB^*-algebra after symmetrizing its product (see Fact 3.3.4); JB^*-algebras and JB-algebras coincide after a categorical correspondence (a fact already noted when we reviewed [762]); non-commutative JB^*-algebras become JB^*-triples in a natural way (see Theorem 4.1.45); and every JB^*-triple can be seen as a closed subtriple of a suitable JB^*-algebra (a fact collected without proof in Theorem 4.1.113). Therefore most basic results in the classical theory of JB^*-algebras, JB-algebras, and JB^*-triples will be involved in our development.

Since JB^*-triples had not appeared when we reviewed [762], let us comment about them briefly. Roughly speaking, JB^*-triples become a functional-analytic solution to the problem of the classification of all 'bounded symmetric domains' in complex Banach spaces. Partial solutions to this problem in the same line are due to Loos [772], who settled the finite-dimensional case, and to Harris [313], who proved that the open unit ball of each norm-closed subspace of any C^*-algebra, which is also closed under the triple product $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$, is a bounded symmetric domain. The definitive solution, due to Kaup [380, 381], asserts that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable complex Banach space, and that if the open unit ball of a complex Banach space is a bounded symmetric domain, then the Banach space itself is *almost* a C^*-algebra, and there is an intrinsically defined triple product $\{\cdots\}$ on it which behaves algebraically and geometrically like the one obtained from the binary product of a C^*-algebra by taking $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$. The precise formulation of this last fact gives rise to the definition of a JB^*-triple (see §2.2.27 and Fact 4.1.41).

JB^*-triples have been intensively studied in recent years and their basic theory can be found in the monographs of Chu [710], Dineen [721], Friedman and Scarr [732], Iordanescu [750], Isidro and Stachó [751], and Upmeier [814, 815], as well as in the survey papers of Chu and Mellon [173], Kaup [384], Rodríguez [525], and Russo [547]. The initial binary approach of our book complements these works.

About the organization of the book

The work we are introducing is covered in two volumes. Roughly speaking, the dividing line between the two can be drawn between what can be done before and after involving the holomorphic theory of JB^*-triples and the structure theory of non-commutative JB^*-algebras. Volume 1 is now concluded, whereas Volume 2 exists today only in the authors' minds. Therefore we are going to describe in detail the content of the first volume (Chapters 1–4), and announce in a less precise form what we intend to do in the second (Chapters 5–8).

Volume 1

In Chapter 1, we develop the basic theory of normed algebras, putting special emphasis on the cases of complete normed unital associative complex algebras and of (associative) C^*-algebras. Non-associative normed algebras are considered here only when they do not offer special difficulties, or difficulties can be overcome in an elementary way. Thus, the first three sections of the chapter are mainly devoted to attracting the attention of the non-expert reader. The chapter is complemented with a fourth section where some selected topics in the theory of compact and weakly compact operators (including recent developments [441, 596]) are discussed.

Chapter 2 is essentially devoted to settling the two first steps in the proof of the 'non-associative Vidav–Palmer theorem' (Theorem VP), namely that the natural involution of any Vidav–Palmer algebra is an algebra involution (see Theorem 2.3.8) and that Vidav–Palmer algebras are non-commutative Jordan algebras (see Theorem 2.4.11). Among the applications of these results, we emphasize the Kadison–Paterson–Sinclair theorem on isometries of C^*-algebras [358, 480] proved in Theorem 2.2.19, the Blecher–Ruan–Sinclair non-associative characterization of (associative) C^*-algebras [106] proved in Theorem 2.4.27, and the non-commutative Urbanik–Wright theorem (already discussed when we reviewed [762]) proved in Theorem 2.6.21. (We missed the formulation and a proof of this last theorem in the delightful book *Numbers* [727].) Applications of the study of contractive projections on C^*-algebras are also made (see Theorems 2.3.68 and 2.4.24). As absolute-valued algebras arise naturally in the Urbanik–Wright theorem, we devote special attention to them (see Sections 2.6, 2.7, and 2.8). Since the Vidav–Palmer axiom depends only on the normed space of the algebra and on the unit, we follow [22, 425] to develop, where possible, the theory of numerical ranges of elements of norm-unital normed algebras in the more general setting of a normed space, in which a norm-one element has been distinguished (see Sections 2.1 and 2.9). This approach (which involves relevant results of pure geometry of normed spaces [56, 268, 287, 291, 293, 299]) complements those of Bonsall–Duncan [694, 695, 696], Doran–Belfi [725], and Palmer [786, 787] in their books.

In Chapter 3 our development depends heavily on the basic theory of *JB*-algebras, which is taken without proof from Hanche-Olsen and Størmer [738], and is complemented in some aspects not covered by that book (see Section 3.1 and Subsection 3.4.1). In particular, surjective linear isometries between *JB*-algebras are described in detail (see Theorem 3.1.21), and Wright's categorical correspondence between *JB*-algebras and JB^*-algebras [641] is established (see Fact 3.4.9). Chapter 3 is essentially devoted to proving Theorem GN (see Theorem 3.2.5), concluding the proof of Theorem VP (see Theorem 3.3.11), developing the theory of alternative C^*-algebras and of non-commutative JB^*-algebras in those aspects which do not involve the so-called 'Jordan spectral theory' (see Subsections 3.4.2, 3.4.4, 3.5.1, 3.5.3, and 3.6.2), and proving the unit-free variant of Theorem GN (see Theorem 3.5.53). The behaviour of the original Gelfand–Naimark axiom $\|a^*a\| = \|a^*\|\|a\|$ in the non-associative setting is fully discussed, generalizing the associative forerunners due to Glimm and Kadison [290] and Vowden [629] (see Subsections 3.5.2, 3.5.4, and 3.5.6). Some auxiliary results, taken from the non-geometric theory of non-associative normed algebras, are also included. Thus Dixmier's fundamental theorem [723] on continuous automorphisms, which are exponentials of continuous derivations, is proved (see Theorem 3.4.49).

In Chapter 4, the Jacobson–McCrimmon notion of the Jordan inverse [754, 433, 436] is involved to derive a spectral theory for normed non-commutative Jordan algebras, which generalizes the one developed in Sections 1.1 and 1.3 for normed associative algebras. This is done in Subsections 4.1.1, 4.1.2, 4.1.4, and 4.1.5. Jordan spectral theory is applied to continue the development of the basic theory of alternative C^*-algebras and of non-commutative JB^*-algebras. In particular, the relationship between non-commutative JB^*-algebras and JB^*-triples is settled (see Theorems 4.1.45 and 4.1.55). The functional-analytic treatment of JB^*-triples is continued in Section 4.2, where Kaup's commutative Gelfand–Naimark type theorems for JB^*-triples [380, 381] are proved (see Theorems 4.2.7 and 4.2.9) and then, following [126, 269, 385, 655], the convexity properties of the closed unit balls of JB^*-triples and of non-commutative JB^*-algebras are established (see Theorems 4.2.24, 4.2.28, 4.2.34, and 4.2.36). Following [77, 78], we describe C^*-algebras and JB^*-algebras generated by a non-self-adjoint idempotent (see Theorems 4.3.11, 4.3.16, 4.3.29, and 4.3.32), and derive Spitkovsky's theorem [595], that C^*-algebras generated by a non-self-adjoint idempotent are generated by two self-adjoint idempotents (see Corollary 4.3.17); we also discuss the appropriate variant of Spitkovsky's theorem for JB^*-algebras (see Corollary 4.3.34). We prove that non-commutative JB^*-algebras have minimum norm topology (see Theorem 4.4.29) and minimality of norm (see Proposition 4.4.34). These results generalize associative forerunners by Cleveland [176] and Bonsall [111], respectively.

We state Behncke's theory of complete normed hermitian Jordan complex $*$-algebras [82], which is proved following the Aupetit–Youngson arguments [48], and is presented in a somewhat new way involving the so-called JB^*-representations (see Theorem 4.5.29 and Corollary 4.5.30). Then a theory of complete normed hermitian alternative complex $*$-algebras is derived (see Theorem 4.5.37 and Corollary 4.5.39) in such a way that it contains the classical associative forerunners [493, 565] as stated in Bonsall–Duncan [696, Section 41]. Generalizing Sakai's theorem [807], we prove that domains of closed densely defined derivations of any unital non-commutative JB^*-algebra are closed under the functional calculus of class C^2 at self-adjoint elements (see Theorem 4.6.63). The chapter also includes some auxiliary results taken from the general theory of non-associative normed algebras. Thus the non-associative generalization [516] of Johnson's uniqueness-of-norm theorem [353] (see also [696, 715, 786]), as well as Aupetit's celebrated forerunner for non-commutative Jordan algebras [40], are settled (see Section 4.4). Along the same lines, a non-associative version of [522], as well as non-associative applications of Bollobás' extremal algebra [110, 182], are discussed (see Section 4.6).

Volume 2

Chapter 5 will be devoted to proving what can be seen as a unit-free version of the non-associative Vidav–Palmer theorem, namely that non-commutative JB^*-algebras are precisely those complete normed complex algebras having an approximate unit bounded by one, and whose open unit ball is a bounded symmetric domain [365]. Some ingredients in the long proof of this result have been already established in Volume 1. This is the case of the Bohnenblust–Karlin Corollary 2.1.13, the non-associative Vidav–Palmer theorem (Theorem 3.3.11) as well as its dual version (Corollary 3.3.26), Proposition 3.5.23, Theorem 4.1.45, and the equivalence (ii)$\Leftrightarrow$(vii) in the Braun–Kaup–Upmeier Theorem 4.2.24. The new relevant ingredients to be proved in the chapter are: (i) the Chu–Iochum–Loupias result that bounded linear operators from a JB^*-triple to its dual are weakly compact [172] (equivalently, via [717, Corollary on p. 12], that all continuous products on the Banach space of a JB^*-triple are Arens regular); (ii) Kaup's theorem that JB^*-triples are precisely those complex Banach spaces whose open unit ball is a bounded symmetric domain [381]; (iii) the contractive projection theorem for JB^*-triples collected without proof in Theorem 2.3.74; and (iv) Dineen's celebrated result that the bidual of a JB^*-triple is a JB^*-triple [213].

Chapter 6 will contain the representation theory for alternative C^*-algebras, already sketched when we reviewed [762], and the representation theory for non-commutative JB^*-algebras, following [19, 124, 222, 481, 482, 641]. In these papers a precise classification of certain prime non-commutative JB^*-algebras (the so-called 'non-commutative JBW^*-factors') is obtained, and the fact that every non-commutative JB^*-algebra has a faithful family of the so-called 'Type I' factor representations is proven. When these results specialize for classical C^*-algebras, Type I non-commutative JBW^*-factors are nothing other than the (associative) W^*-factors consisting of all bounded linear operators on some complex Hilbert space [738, Proposition 7.5.2], and, consequently, Type I factor representations are precisely irreducible representations on Hilbert spaces. The chapter will contain also a classification of all prime non-commutative JB^*-algebras, obtained in [255, 363] by applying Zel'manov's purely algebraic techniques in [437, 662, 663]. Many applications of the representation theory will be discussed. Among them, we emphasize the generalization to non-commutative JB^*-algebras [340] of Kaplansky's characterization of commutativity of C^*-algebras by means of the absence of isotropic elements [761, Theorem B in Appendix III].

In Chapters 7 and 8, we will discuss selected topics in the theory of non-associative normed algebras, which need not be directly related to the non-associative generalizations of C^*-algebras. Chapter 7 will deal with the analytic treatment of Zel'manov's prime theorems for Jordan structures, already sketched in Chapter 6. Thus several direct contributions of Zel'manov to the theory of normed Jordan algebras [147, 538] (one of which was refined later in [447]) and the binary results in [146, 151, 152, 539] will be included with proofs. The ternary results in [448, 449] will be simply surveyed. The survey papers [145, 450, 532] could provide the reader with a more detailed overview of the intended content of the whole of Chapter 7.

The concluding Chapter 8 will deal with miscellany in the theory of non-associative normed algebras. We will complement our knowledge on non-associative generalizations of Rickart's dense-range-homomorphism theorem (see Theorem 4.1.19 and Proposition 4.1.108) with those obtained in [165, 529]. Complementing Corollary 4.4.55, some automatic continuity theorems for homomorphisms 'into', taken from [165, 462, 529], will also be included. Automatic continuity of Lie homomorphisms, culminating in [130] through the papers of Berenguer–Villena [97, 98] and Aupetit–Mathieu [47] already discussed in Subsection 4.4.5, will also receive special attention. As an auxiliary tool for the proof of the main result in [130], we will incorporate the discussion in [91] about normed Jordan algebras 'with finite spectrum'. Actually, the theory of normed Jordan structures subjected to 'finiteness conditions' would merit being systematically organized. Nevertheless, we will not be doing this, and will limit ourselves to surveying this matter by reviewing results from Aupetit [43, 44], Aupetit–Baribeau [45], Aupetit–Maouche [46], Benslimane–Boudi [87, 88], Benslimane–Fernández–Kaidi [89], Benslimane–Jaa–Kaidi [90], Benslimane–Kaidi [91], Benslimane–Rodríguez [95], Boudi [119], Boudi–Marhnine–Zarhouti–Fernández–García [120], Bouhya–Fernández [121], Fernández [250, 251, 252], Fernández–García–Sánchez [256], Fernández–Rodríguez [258], Hessenberger [322, 323, 324, 325], Hessenberger–Maouche [326], Loos [402, 404, 405], Maouche [413, 412], Pérez–Rico–Rodríguez–Villena [489], and Wilkins [636]. Another favourite topic to be included in this chapter is that of the general theory of non-associative H^*-algebras, which incidentally appears in Remark 2.6.54, Lemma 2.7.50, Subsection 2.8.2, and Corollary 4.1.104 of this volume. This will be done by taking the appropriate material from the papers [142, 144, 148, 149, 198, 199, 259, 526, 624] (see also [687, Chapters 7 and 8] and [525, Section E]). The chapter will conclude with the non-associative discussion of the Rota–Strang paper [544] covered in [452, 453].

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