

Contents

<i>Preface</i>	<i>page xi</i>
1 Introduction to difference equations	1
1.1 A first look at discrete equations	1
1.2 The Riccati equation	16
1.3 Partial difference equations	21
1.4 Notes	27
Exercises	28
2 Discrete equations from transformations of continuous equations	31
2.1 Special functions and linear equations	32
2.2 Addition formulae	41
2.3 The Painlevé equations	44
2.4 Bäcklund transformations for nonlinear PDEs	50
2.5 Infinite sequence of conservation laws and KdV hierarchy	53
2.6 Notes	61
Exercises	62
3 Integrability of PΔEs	67
3.1 Quadrilateral PΔEs	68
3.2 Consistency-around-the-cube as integrability	71
3.3 Lax pairs and Bäcklund transformation from CAC	75
3.4 Yang–Baxter maps	82
3.5 Classification of quadrilateral PΔEs	86
3.6 Different equations on different faces of the consistency cube	90
3.7 CAC for multi-component equations	94
3.8 Lattice KdV, SKdV and mKdV equations	103
3.9 Higher-dimensional equations: the KP class	110
3.10 Notes	114
Exercises	116
	vii

viii	<i>Contents</i>	
4	Interlude: Lattice equations and numerical algorithms	119
4.1	Padé approximants	119
4.2	Convergence acceleration algorithm	126
4.3	Rutishauser's QD algorithm	128
4.4	Notes	133
	Exercises	134
5	Continuum limits of lattice PΔE	136
5.1	How to take a continuum limit	136
5.2	Plane-wave factors and linearization	137
5.3	The semi-continuous limits	139
5.4	Semi-discrete Lax pairs	144
5.5	Full continuum limit	147
5.6	All at once, or the double continuum limit	151
5.7	Continuum limits of the 9-point BSQ	152
5.8	Notes	154
	Exercises	155
6	One-dimensional lattices and maps	159
6.1	Integrability of maps	159
6.2	The Kahan–Hirota–Kimura discretization	168
6.3	The QRT maps	169
6.4	Periodic reductions	177
6.5	Lax pair for the periodic reductions and construction of invariants	185
6.6	Pole reduction of the semi-discrete KP equation	189
6.7	Notes	193
	Exercises	194
7	Identifying integrable difference equations	197
7.1	Singularity analysis of differential and difference equations	197
7.2	Algebraic entropy	207
7.3	Singularities from a geometric point of view	213
7.4	Notes	219
	Exercises	221
8	Hirota's bilinear method	223
8.1	Introduction	223
8.2	Soliton solutions	226
8.3	Hirota's and Miwa's equations	229
8.4	Reductions of the Hirota–Miwa equation	233
8.5	Bilinearization of a lattice equation	238

<i>Contents</i>		ix
8.6	Solutions in matrix form	241
8.7	Notes	245
	Exercises	246
9	Multi-soliton solutions and the Cauchy matrix scheme	250
9.1	Cauchy matrix structure for KdV-type equations	250
9.2	Closed-form lattice equations	255
9.3	Derivation of Lax pairs	257
9.4	Bilinear form from soliton solutions	261
9.5	The NQC and Q3 equations	266
9.6	Proof of the Q3 N -soliton solution	268
9.7	Higher-dimensional soliton systems: the KP class	272
9.8	Notes	277
	Exercises	277
10	Similarity reductions of integrable PΔEs	280
10.1	Introduction to dimensional reductions	280
10.2	Compatibility of lattice constraint with quad equations	284
10.3	The linear case	285
10.4	Similarity constraints for the lattice KdV family	289
10.5	Notes	300
	Exercises	301
11	Discrete Painlevé equations	304
11.1	Early discoveries of discrete Painlevé equations	305
11.2	Discrete Painlevé equations from Sakai's classification	307
11.3	Coalescences and degeneracies of the discrete Painlevé equations	310
11.4	Bäcklund and other transformations of discrete Painlevé equations	311
11.5	Affine Weyl groups	315
11.6	Linear problems	322
11.7	Linearization of discrete Painlevé equations	326
11.8	Sakai's elliptic discrete Painlevé equation	328
11.9	Notes	329
	Exercises	331
12	Lagrangian multiform theory	334
12.1	Conventional Lagrange theory and its discrete analogue	335
12.2	Lagrangian 2-form structure	343
12.3	Lagrangian 1-form structure	352
12.4	Notes	359
	Exercises	360

<i>Appendix A</i>	<i>Elementary difference calculus and difference equations</i>	363
<i>Appendix B</i>	<i>Theta functions and elliptic functions</i>	384
<i>Appendix C</i>	<i>The continuous Painlevé equations and the Garnier system</i>	404
<i>Appendix D</i>	<i>Some determinantal identities</i>	407
<i>References</i>		411
<i>Index</i>		440