

## DISCRETE SYSTEMS AND INTEGRABILITY

This first introductory text to discrete integrable systems introduces key notions of integrability from the vantage point of discrete systems, also making connections with the continuous theory where relevant. While treating the material at an elementary level, the book also highlights many recent developments. Topics include: Darboux and Bäcklund transformations, difference equations and special functions, multidimensional consistency of integrable lattice equations, associated linear problems (Lax pairs), connections with Padé approximants and convergence algorithms, singularities and geometry, Hirota's bilinear formalism for lattices, intriguing properties of discrete Painlevé equations and the novel theory of Lagrangian multiforms. The book builds the material in an organic way, emphasizing interconnections between the various approaches, while the exposition is mostly done through explicit computations on key examples. Written by respected experts in the field, the book's numerous exercises and thorough list of references will benefit both upper-level undergraduate and beginning graduate students as well as researchers from other disciplines.

JARMO HIETARINTA is Professor Emeritus of Theoretical Physics at the University of Turku, Finland. His work has focused on the search for integrable systems of various forms, including Hamiltonian mechanics, Hirota bilinear form, Yang–Baxter and tetrahedron equations as well as lattice equations. He was instrumental in the setting up of the nlin.SI category in arxiv.org and created the web pages for the SIDE (Symmetries and Integrability of Difference Equations) conference series: <http://side-conferences.net>

NALINI JOSHI is Professor of Applied Mathematics at the University of Sydney. She is best known for her work on the Painlevé equations and works at the leading edge of international efforts to analyze discrete and continuous integrable systems in the geometric setting of their initial-value spaces, constructed by resolving singularities in complex projective space. She was elected as a Fellow of the Australian Academy of Science in 2008, holds a Georgina Sweet Australian Laureate Fellowship and was awarded the special Hardy Fellowship of the London Mathematical Society in 2015.

FRANK NIJHOFF is Professor of Mathematical Physics in the School of Mathematics of the University of Leeds. His research focuses on nonlinear difference and differential equations, symmetries and integrability of discrete systems, variational calculus, quantum integrable systems and linear and nonlinear special functions. He was the principal organizer of the 2009 six-month programme on Discrete Integrable Systems at the Isaac Newton Institute, and a Royal Society Leverhulme Trust Senior Research Fellow in 2011.

## Cambridge Texts in Applied Mathematics

All titles listed below can be obtained from good booksellers or from Cambridge University Press.  
For a complete series listing, visit [www.cambridge.org/mathematics](http://www.cambridge.org/mathematics).

*Nonlinear Dispersive Waves*

MARK J. ABLOWITZ

*Flow, Deformation and Fracture*

G. I. BARENBLATT

*Hydrodynamic Instabilities*

FRANÇOIS CHARRU

*The Mathematics of Signal Processing*

STEVEN B. DAMELIN & WILLARD MILLER, JR

*An Introduction to Stochastic Dynamics*

JINQIAO DUAN

*Singularities: Formation, Structure and Propagation*

J. EGGERS & M. A. FONTELOS

*A First Course in Continuum Mechanics*

OSCAR GONZALEZ & ANDREW M. STUART

*A Physical Introduction to Suspension Dynamics*

ÉLISABETH GUAZZELLI & JEFFREY F. MORRIS

*Applied Solid Mechanics*

PETER HOWELL, GREGORY KOZYREFF & JOHN OCKENDON

*A First Course in the Numerical Analysis of Differential Equations (2nd Edition)*

ARIEH ISERLES

*Iterative Methods in Combinatorial Optimization*

LAP CHI LAU, R. RAVI & MOHIT SINGH

*An Introduction to Polynomial and Semi-Algebraic Optimization*

JEAN BERNARD LASSERRE

*An Introduction to Computational Stochastic PDEs*

GABRIEL J. LORD, CATHERINE E. POWELL & TONY SHARDLOW

# DISCRETE SYSTEMS AND INTEGRABILITY

J. HIETARINTA

*University of Turku*

N. JOSHI

*University of Sydney*

F. W. NIJHOFF

*University of Leeds*



CAMBRIDGE  
UNIVERSITY PRESS

CAMBRIDGE  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107042728](http://www.cambridge.org/9781107042728)

© Cambridge University Press 2016

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2016

Printed in the United Kingdom by Clays, St Ives plc

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*

Names: Hietarinta, J. (Jarmo), author. | Joshi, Nalini, author. | Nijhoff, Frank W., author.

Title: Discrete systems and integrability / J. Hietarinta (University of Turku), N. Joshi (University of Sydney), F.W. Nijhoff (University of Leeds).

Other titles: Cambridge texts in applied mathematics.

Description: Cambridge, United Kingdom ; New York, NY : Cambridge University Press, 2016. | © 2016 | Series: Cambridge texts in applied mathematics | Includes bibliographical references and index.

Identifiers: LCCN 2016005109 | ISBN 9781107042728 (hardback) | ISBN 1107042720 (hardback) | ISBN 9781107669482 (paperback) | ISBN 1107669480 (paperback)

Subjects: LCSH: Integral equations. | Mathematical physics.

Classification: LCC QC20.7.I58 H54 2016 | DDC 511/.1-dc23

LC record available at <http://lccn.loc.gov/2016005109>

ISBN 978-1-107-04272-8 Hardback

ISBN 978-1-107-66948-2 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

To Marja, Robert and Alain

## Contents

<i>Preface</i>	<i>page xi</i>
<b>1 Introduction to difference equations</b>	<b>1</b>
1.1 A first look at discrete equations	1
1.2 The Riccati equation	16
1.3 Partial difference equations	21
1.4 Notes	27
Exercises	28
<b>2 Discrete equations from transformations of continuous equations</b>	<b>31</b>
2.1 Special functions and linear equations	32
2.2 Addition formulae	41
2.3 The Painlevé equations	44
2.4 Bäcklund transformations for nonlinear PDEs	50
2.5 Infinite sequence of conservation laws and KdV hierarchy	53
2.6 Notes	61
Exercises	62
<b>3 Integrability of PΔEs</b>	<b>67</b>
3.1 Quadrilateral PΔEs	68
3.2 Consistency-around-the-cube as integrability	71
3.3 Lax pairs and Bäcklund transformation from CAC	75
3.4 Yang–Baxter maps	82
3.5 Classification of quadrilateral PΔEs	86
3.6 Different equations on different faces of the consistency cube	90
3.7 CAC for multi-component equations	94
3.8 Lattice KdV, SKdV and mKdV equations	103
3.9 Higher-dimensional equations: the KP class	110
3.10 Notes	114
Exercises	116
	vii

<b>4</b>	<b>Interlude: Lattice equations and numerical algorithms</b>	119
4.1	Padé approximants	119
4.2	Convergence acceleration algorithm	126
4.3	Rutishauser's QD algorithm	128
4.4	Notes	133
	Exercises	134
<b>5</b>	<b>Continuum limits of lattice PΔE</b>	136
5.1	How to take a continuum limit	136
5.2	Plane-wave factors and linearization	137
5.3	The semi-continuous limits	139
5.4	Semi-discrete Lax pairs	144
5.5	Full continuum limit	147
5.6	All at once, or the double continuum limit	151
5.7	Continuum limits of the 9-point BSQ	152
5.8	Notes	154
	Exercises	155
<b>6</b>	<b>One-dimensional lattices and maps</b>	159
6.1	Integrability of maps	159
6.2	The Kahan–Hirota–Kimura discretization	168
6.3	The QRT maps	169
6.4	Periodic reductions	177
6.5	Lax pair for the periodic reductions and construction of invariants	185
6.6	Pole reduction of the semi-discrete KP equation	189
6.7	Notes	193
	Exercises	194
<b>7</b>	<b>Identifying integrable difference equations</b>	197
7.1	Singularity analysis of differential and difference equations	197
7.2	Algebraic entropy	207
7.3	Singularities from a geometric point of view	213
7.4	Notes	219
	Exercises	221
<b>8</b>	<b>Hirota's bilinear method</b>	223
8.1	Introduction	223
8.2	Soliton solutions	226
8.3	Hirota's and Miwa's equations	229
8.4	Reductions of the Hirota–Miwa equation	233
8.5	Bilinearization of a lattice equation	238

*Contents*

ix

8.6	Solutions in matrix form	241
8.7	Notes	245
	Exercises	246
<b>9</b>	<b>Multi-soliton solutions and the Cauchy matrix scheme</b>	<b>250</b>
9.1	Cauchy matrix structure for KdV-type equations	250
9.2	Closed-form lattice equations	255
9.3	Derivation of Lax pairs	257
9.4	Bilinear form from soliton solutions	261
9.5	The NQC and Q3 equations	266
9.6	Proof of the Q3 $N$ -soliton solution	268
9.7	Higher-dimensional soliton systems: the KP class	272
9.8	Notes	277
	Exercises	277
<b>10</b>	<b>Similarity reductions of integrable PΔEs</b>	<b>280</b>
10.1	Introduction to dimensional reductions	280
10.2	Compatibility of lattice constraint with quad equations	284
10.3	The linear case	285
10.4	Similarity constraints for the lattice KdV family	289
10.5	Notes	300
	Exercises	301
<b>11</b>	<b>Discrete Painlevé equations</b>	<b>304</b>
11.1	Early discoveries of discrete Painlevé equations	305
11.2	Discrete Painlevé equations from Sakai's classification	307
11.3	Coalescences and degeneracies of the discrete Painlevé equations	310
11.4	Bäcklund and other transformations of discrete Painlevé equations	311
11.5	Affine Weyl groups	315
11.6	Linear problems	322
11.7	Linearization of discrete Painlevé equations	326
11.8	Sakai's elliptic discrete Painlevé equation	328
11.9	Notes	329
	Exercises	331
<b>12</b>	<b>Lagrangian multiform theory</b>	<b>334</b>
12.1	Conventional Lagrange theory and its discrete analogue	335
12.2	Lagrangian 2-form structure	343
12.3	Lagrangian 1-form structure	352
12.4	Notes	359
	Exercises	360

<i>Appendix A</i>	<i>Elementary difference calculus and difference equations</i>	363
<i>Appendix B</i>	<i>Theta functions and elliptic functions</i>	384
<i>Appendix C</i>	<i>The continuous Painlevé equations and the Garnier system</i>	404
<i>Appendix D</i>	<i>Some determinantal identities</i>	407
<i>References</i>		411
<i>Index</i>		440

## Preface

There has been a surge of interest in discrete integrable systems in the last two decades. The term “discrete integrable systems” (DIS) combines two aspects: discreteness and integrability. The subtle concept of integrability touches on global existence and regularity of solutions, explicit solvability, as well as compatibility and consistency – fundamental features, which form the recurrent themes of this book. On the other hand, by discrete systems we mean mathematical models that involve *finite* (as opposed to *infinitesimal*) operations. In a sense, discrete systems are essential to an understanding of integrability, and this book serves to provide an introduction to integrability from the perspective of discrete systems.

Discrete integrable systems include many types of equations, such as recurrence relations, difference equations and dynamical mappings as well as equations that contain a mixture of derivative and difference operators. Integrable systems have appeared throughout the history of mathematics without being recognized as integrable. An example is the equation that arises from the geometric collinearity theorem of Menelaus of Alexandria in the second century. Other examples are found in the defining equations of classical and nonclassical special functions. Important physical models, such as the equations of motion of the Euler top, are also integrable. The elliptic billiard is a classic example of a DIS, and the corresponding geometric result is the *Poncelet’s porism*. It is, however, only in the second half of the twentieth century that the subject of integrable systems has grown as a discipline in its own right, and only in the last roughly two decades that the field of DIS has come to prominence as an area in which numerous breakthroughs have taken place, inspiring new developments in other areas of mathematics.

The number of integrable systems is large and growing, and the list of them includes a large number of examples that have application to physics and other scientific fields. In many cases, it turns out that one and the same discrete equation may be interpreted in different ways: as a dynamical map, as a difference equation and as an addition formula. This means that in the study of a given DIS a large number of branches of mathematics come together. Moreover, integrable equations are not necessarily isolated objects, but have many mathematical interconnections between each other. Highlighting these interconnections is another aim of this book.

Integrable systems are both rare and universal. Like prime numbers, their rarity is due to the intricate mathematical structures underlying them, while at the same time these structures explain their universality. This book aims to describe their underlying structures, but there are still many that remain to be discovered.

This book grew from lectures given by the authors in courses on DIS for students at the advanced undergraduate or beginning graduate level, at the University of Leeds, the University of Turku and the University of Sydney. The presentation of the book is pitched at a level suitable for students as well as researchers who are seeking an introduction to the subject.

The exposition of the book will be led by exploring key examples. We will show how the major features of integrability arise in an organic way by revealing them through explicit calculations. Each chapter contains several exercises, which will further illustrate the methods and expand on the main text. The initial chapters contain the basic material. Additional specialized or advanced topics are provided in the later chapters. Some important background material on difference calculus, elliptic functions and determinantal identities are provided in the appendices.

We start in Chapter 1 with an overview of the types of difference equations we will encounter, highlighting integrable systems as examples. In Chapter 2 we show how discrete systems, in the form of difference equations, arise naturally from known continuous equations in two different settings: the linear case (with difference equations arising from the recurrence structure of families of special functions) and the nonlinear case (addition rules for elliptic functions and Bäcklund–Schlesinger transformations for Painlevé transcendents). In the multivariable case we show how integrable partial difference equations arise as permutability conditions from Bäcklund transforms of soliton equations. This emergence of fully discrete equations (also called *lattice equations*) leads naturally to a characterization of integrability by *multidimensional consistency*. This property is explored in depth in Chapter 3, where we study its consequences and also discuss additional sources of DIS. Chapter 4 provides an interlude from the main theory by illustrating the links between integrable systems and applications in the theory of numerical algorithms. The circle is closed in Chapter 5 by providing continuum limits that lead back to continuous systems from the discrete ones. In the remaining chapters we will delve deeper into the theory and explore reductions and solutions, as well as other characteristics of integrability. In Chapter 6, we shift the focus to ordinary difference equations and dynamical mappings and provide various points of view to study them. The problem of detecting and identifying integrable difference equations by methods based on their singularity structure and growth is discussed in Chapter 7. Chapters 8 and 9 discuss two different approaches to the construction of special solutions of the partial difference equations encountered in the earlier chapters. First, in Chapter 8, we present and explain Hirota’s bilinear method: in particular, how it can be used to obtain soliton solutions. In Chapter 9, another approach to soliton solutions is provided by the Cauchy matrix approach, which allows us to see that many seemingly different systems are closely interrelated. Other types of solutions are obtained in Chapter 10 through similarity reductions of partial difference equations. These

are transcendental solutions obtained as solutions of a class of nonautonomous nonlinear ordinary difference equations, called the discrete Painlevé equations. The discrete Painlevé equations have very deep mathematical properties, which are described in Chapter 11. The final chapter, Chapter 12, provides insight into the geometrical aspects of integrable systems through their Lagrangian structure and the corresponding new variational approach suitable for integrable systems.

The book is by no means intended to be an exhaustive treatment of all topics related to discrete integrability, but rather reflects the authors' interests and experience. For example, we do not give a comprehensive treatment of Lie symmetries and conservation laws of discrete systems; for that, we refer to Hydon (2014). Other topics we omit are ultra-discrete equations and tropical geometry, as well as discrete quantum integrable systems. Currently, there exist only a few monographs on DIS: Suris (2003) focuses primarily on integrable discretizations from a Hamiltonian perspective, while Bobenko and Suris (2008) consider differential geometric aspects; Duistermaat (2010) deals mainly with the QRT (Quispel–Roberts–Thompson) dynamical mapping and the algebraic geometry of the associated elliptic surfaces.

We would like to draw the reader's attention also to the biennial conference series SIDE, devoted to *Symmetries and Integrability of Difference Equations*, the proceedings of which provide a good account of the state of the art of the subject. For further details, see <http://side-conferences.net>. We would also like to mention some collections of lectures on DIS; for example, Grammaticos et al. (2004) and Levi et al. (2011).

Many colleagues and students have contributed to the evolution of this book in various ways. We would in particular like to thank (in alphabetical order) James Atkinson, Sam Butler, Neslihan Delice, Chris Field, Wei Fu, Basil Grammaticos, Mike Hay, Anthony Henderson, Phil Howes, Paul Jennings, Pavlos Kassotakis, Sotiris Konstantinou-Rizos, Sarah Lobb, Nobutaka Nakazono, Maciej Nieszporski, Chris Ormerod, Vassilis Papageorgiou, Reinout Quispel, Alfred Ramani, John Roberts, Yang Shi, Ying Shi, Paul Spicer, Tomoyuki Takenawa, Tasos Tongas, Dinh Tran, Peter van der Kamp, Claude Viallet, Pavlos Xenitidis, Sikarin Yoo-Kong, Da-jun Zhang and Songlin Zhao. Special thanks go to Da-jun Zhang for his careful and detailed reading of the galley proofs.