Introduction to Malliavin Calculus

This textbook offers a compact introductory course on Malliavin calculus, an active and powerful area of research. It covers recent applications including density formulas, regularity of probability laws, central and noncentral limit theorems for Gaussian functionals, convergence of densities, and noncentral limit theorems for the local time of Brownian motion. The book also includes self-contained presentations of Brownian motion and stochastic calculus as well as of Lévy processes and stochastic calculus for jump processes. Accessible to nonexperts, the book can be used by graduate students and researchers to develop their mastery of the core techniques necessary for further study.

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To my wife, Maria Pilar

To my daughter, Juliette

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Preface

This textbook provides an introductory course on Malliavin calculus intended to prepare the interested reader for further study of existing monographs on the subject such as Bichteler *et al.* (1987), Malliavin (1991), Sanz-Solé (2005), Malliavin and Thalmaier (2005), Nualart (2006), Di Nunno *et al.* (2009), Nourdin and Peccati (2012), and Ishikawa (2016), among others. Moreover, it contains recent applications of Malliavin calculus, including density formulas, central limit theorems for functionals of Gaussian processes, theorems on the convergence of densities, noncentral limit theorems, and Malliavin calculus for jump processes. Recommended prior knowledge would be an advanced probability course that includes laws of large numbers and central limit theorems, martingales, and Markov processes.

The Malliavin calculus is an infinite-dimensional differential calculus on Wiener space, first introduced by Paul Malliavin in the 1970s with the aim of giving a probabilistic proof of Hörmander's hypoellipticity theorem; see Malliavin (1978a, b, c). The theory was further developed, see e.g. Shigekawa (1980), Bismut (1981), Stroock (1981a, b), and Ikeda and Watanabe (1984), and since then many new applications have appeared.

Chapters 1 and 2 give an introduction to stochastic calculus with respect to Brownian motion, as developed by Itô (1944). The purpose of this calculus is to construct stochastic integrals for adapted and square integrable processes and to develop a change-of-variable formula.

Chapters 3, 4, and 5 present the main operators of the Malliavin calculus, which are the derivative, the divergence, the generator of the Ornstein–Uhlenbeck semigroup, and the corresponding Sobolev norms. In Chapter 4, multiple stochastic integrals are constructed following Itô (1951), and the orthogonal decomposition of square integrable random variables due to Wiener (1938) is derived. These concepts play a key role in the development of further properties of the Malliavin calculus operators. In particular, Chapter 5 contains an integration-by-parts formula that relates the three operators, which is crucial for applications. In particular, it allows us to prove a density formula due to Nourdin and Viens (2009).

Chapters 6, 7, and 8 are devoted to different applications of the Malliavin calculus for Brownian motion. Chapter 6 presents two different stochastic integral representations: the first is the well-known Clark–Ocone formula, and the second uses the inverse of the Ornstein–Ulhenbeck generator. We present, as a consequence of the Clark–Ocone formula, a central limit theorem for the modulus of continuity of the local time of Brownian motion, proved by Hu and Nualart (2009). As an application of the second representation formula, we show how to derive tightness in the asymptotic behavior of the self-intersection local time of fractional Brownian motion, following Hu and Nualart (2005) and Jaramillo and Nualart (2018). In Chapter 7 we develop the Malliavin calculus to derive explicit formulas for the densities of random variables and criteria for their regularity. We apply these criteria to the proof of Hörmander's hypoellipticity theorem. Chapter 8 presents an application of Malliavin calculus, combined with Stein's method, to normal approximations.

Chapters 9, 10, and 11 develop Malliavin calculus for Poisson random measures. Specifically, Chapter 9 introduces stochastic integration for jump processes, as well as the Wiener chaos decomposition of a Poisson random measure. Then the Malliavin calculus is developed in two different directions. In Chapter 10 we introduce the three Malliavin operators and their Sobolev norms using the Wiener chaos decomposition. As an application, we present the Clark–Ocone formula and Stein's method for Poisson functionals. In Chapter 11 we use the theory of cylindrical functionals to introduce the derivative and divergence operators. This approach allows us to obtain a criterion for the existence of densities, which we apply to diffusions with jumps.

Finally, in the appendix we review basic results on stochastic processes that are used throughout the book.