

## Introduction to Malliavin Calculus

This textbook offers a compact introductory course on Malliavin calculus, an active and powerful area of research. It covers recent applications including density formulas, regularity of probability laws, central and noncentral limit theorems for Gaussian functionals, convergence of densities, and noncentral limit theorems for the local time of Brownian motion. The book also includes self-contained presentations of Brownian motion and stochastic calculus as well as of Lévy processes and stochastic calculus for jump processes. Accessible to nonexperts, the book can be used by graduate students and researchers to develop their mastery of the core techniques necessary for further study.

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# Introduction to Malliavin Calculus

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*To my wife, Maria Pilar*

*To my daughter, Juliette*

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## Contents

<i>Preface</i>	<i>page xi</i>
<b>1 Brownian Motion</b>	<b>1</b>
1.1 Preliminaries and Notation	1
1.2 Definition and Basic Properties	1
1.3 Wiener Integral	7
1.4 Wiener Space	9
1.5 Brownian Filtration	9
1.6 Markov Property	10
1.7 Martingales Associated with Brownian Motion	11
1.8 Strong Markov Property	14
Exercises	16
<b>2 Stochastic Calculus</b>	<b>18</b>
2.1 Stochastic Integrals	18
2.2 Indefinite Stochastic Integrals	23
2.3 Integral of General Processes	28
2.4 Itô's Formula	30
2.5 Tanaka's Formula	35
2.6 Multidimensional Version of Itô's Formula	38
2.7 Stratonovich Integral	40
2.8 Backward Stochastic Integral	41
2.9 Integral Representation Theorem	42
2.10 Girsanov's Theorem	44
Exercises	47
<b>3 Derivative and Divergence Operators</b>	<b>50</b>
3.1 Finite-Dimensional Case	50
3.2 Malliavin Derivative	51
3.3 Sobolev Spaces	53
3.4 The Divergence as a Stochastic Integral	56

viii	<i>Contents</i>	
3.5	Isonormal Gaussian Processes	57
	Exercises	61
<b>4</b>	<b>Wiener Chaos</b>	63
4.1	Multiple Stochastic Integrals	63
4.2	Derivative Operator on the Wiener Chaos	65
4.3	Divergence on the Wiener Chaos	68
4.4	Directional Derivative	69
	Exercises	72
<b>5</b>	<b>Ornstein–Uhlenbeck Semigroup</b>	74
5.1	Mehler’s Formula	74
5.2	Generator of the Ornstein–Uhlenbeck Semigroup	78
5.3	Meyer’s Inequality	80
5.4	Integration-by-Parts Formula	83
5.5	Nourdin–Viens Density Formula	84
	Exercises	86
<b>6</b>	<b>Stochastic Integral Representations</b>	87
6.1	Clark–Ocone formula	87
6.2	Modulus of Continuity of the Local Time	90
6.3	Derivative of the Self-Intersection Local Time	96
6.4	Application of the Clark–Ocone Formula in Finance	97
6.5	Second Integral Representation	99
6.6	Proving Tightness Using Malliavin Calculus	100
	Exercises	103
<b>7</b>	<b>Study of Densities</b>	105
7.1	Analysis of Densities in the One-Dimensional Case	105
7.2	Existence and Smoothness of Densities for Random Vectors	108
7.3	Density Formula using the Riesz Transform	111
7.4	Log-Likelihood Density Formula	113
7.5	Malliavin Differentiability of Diffusion Processes	118
7.6	Absolute Continuity under Ellipticity Conditions	122
7.7	Regularity of the Density under Hörmander’s Conditions	123
	Exercises	129
<b>8</b>	<b>Normal Approximations</b>	131
8.1	Stein’s Method	131
8.2	Stein Meets Malliavin	136
8.3	Normal Approximation on a Fixed Wiener Chaos	138
8.4	Chaotic Central Limit Theorem	143
8.5	Applications to Fractional Brownian Motion	146

<i>Contents</i>		ix
8.6	Convergence of Densities	150
8.7	Noncentral Limit Theorems	153
	Exercises	156
<b>9</b>	<b>Jump Processes</b>	158
9.1	Lévy Processes	158
9.2	Poisson Random Measures	160
9.3	Integral with respect to a Poisson Random Measure	163
9.4	Stochastic Integrals with respect to the Jump Measure of a Lévy Process	164
9.5	Itô's Formula	168
9.6	Integral Representation Theorem	172
9.7	Girsanov's Theorem	174
9.8	Multiple Stochastic Integrals	175
9.9	Wiener Chaos for Poisson Random Measures	177
	Exercises	180
<b>10</b>	<b>Malliavin Calculus for Jump Processes I</b>	182
10.1	Derivative Operator	182
10.2	Divergence Operator	187
10.3	Ornstein–Uhlenbeck Semigroup	191
10.4	Clark–Ocone Formula	192
10.5	Stein's Method for Poisson Functionals	193
10.6	Normal Approximation on a Fixed Chaos	194
	Exercises	199
<b>11</b>	<b>Malliavin Calculus for Jump Processes II</b>	201
11.1	Derivative Operator	201
11.2	Sobolev Spaces	205
11.3	Directional Derivative	208
11.4	Application to Diffusions with Jumps	212
	Exercises	220
<b>Appendix A</b>	<b>Basics of Stochastic Processes</b>	221
A.1	Stochastic Processes	221
A.2	Gaussian Processes	222
A.3	Equivalent Processes	223
A.4	Regularity of Trajectories	223
A.5	Markov Processes	223
A.6	Stopping Times	224
A.7	Martingales	225
	<i>References</i>	228
	<i>Index</i>	235



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## Preface

This textbook provides an introductory course on Malliavin calculus intended to prepare the interested reader for further study of existing monographs on the subject such as Bichteler *et al.* (1987), Malliavin (1991), Sanz-Solé (2005), Malliavin and Thalmaier (2005), Nualart (2006), Di Nunno *et al.* (2009), Nourdin and Peccati (2012), and Ishikawa (2016), among others. Moreover, it contains recent applications of Malliavin calculus, including density formulas, central limit theorems for functionals of Gaussian processes, theorems on the convergence of densities, noncentral limit theorems, and Malliavin calculus for jump processes. Recommended prior knowledge would be an advanced probability course that includes laws of large numbers and central limit theorems, martingales, and Markov processes.

The Malliavin calculus is an infinite-dimensional differential calculus on Wiener space, first introduced by Paul Malliavin in the 1970s with the aim of giving a probabilistic proof of Hörmander's hypoellipticity theorem; see Malliavin (1978a, b, c). The theory was further developed, see e.g. Shigekawa (1980), Bismut (1981), Stroock (1981a, b), and Ikeda and Watanabe (1984), and since then many new applications have appeared.

Chapters 1 and 2 give an introduction to stochastic calculus with respect to Brownian motion, as developed by Itô (1944). The purpose of this calculus is to construct stochastic integrals for adapted and square integrable processes and to develop a change-of-variable formula.

Chapters 3, 4, and 5 present the main operators of the Malliavin calculus, which are the derivative, the divergence, the generator of the Ornstein–Uhlenbeck semigroup, and the corresponding Sobolev norms. In Chapter 4, multiple stochastic integrals are constructed following Itô (1951), and the orthogonal decomposition of square integrable random variables due to Wiener (1938) is derived. These concepts play a key role in the development of further properties of the Malliavin calculus operators. In particular, Chapter 5 contains an integration-by-parts formula that relates the three op-

erators, which is crucial for applications. In particular, it allows us to prove a density formula due to Nourdin and Viens (2009).

Chapters 6, 7, and 8 are devoted to different applications of the Malliavin calculus for Brownian motion. Chapter 6 presents two different stochastic integral representations: the first is the well-known Clark–Ocone formula, and the second uses the inverse of the Ornstein–Uhlenbeck generator. We present, as a consequence of the Clark–Ocone formula, a central limit theorem for the modulus of continuity of the local time of Brownian motion, proved by Hu and Nualart (2009). As an application of the second representation formula, we show how to derive tightness in the asymptotic behavior of the self-intersection local time of fractional Brownian motion, following Hu and Nualart (2005) and Jaramillo and Nualart (2018). In Chapter 7 we develop the Malliavin calculus to derive explicit formulas for the densities of random variables and criteria for their regularity. We apply these criteria to the proof of Hörmander’s hypoellipticity theorem. Chapter 8 presents an application of Malliavin calculus, combined with Stein’s method, to normal approximations.

Chapters 9, 10, and 11 develop Malliavin calculus for Poisson random measures. Specifically, Chapter 9 introduces stochastic integration for jump processes, as well as the Wiener chaos decomposition of a Poisson random measure. Then the Malliavin calculus is developed in two different directions. In Chapter 10 we introduce the three Malliavin operators and their Sobolev norms using the Wiener chaos decomposition. As an application, we present the Clark–Ocone formula and Stein’s method for Poisson functionals. In Chapter 11 we use the theory of cylindrical functionals to introduce the derivative and divergence operators. This approach allows us to obtain a criterion for the existence of densities, which we apply to diffusions with jumps.

Finally, in the appendix we review basic results on stochastic processes that are used throughout the book.