Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

1

Brownian Motion

In this chapter we introduce Brownian motion and study several aspects of this stochastic process, including the regularity of sample paths, quadratic variation, Wiener stochastic integrals, martingales, Markov properties, hitting times, and the reflection principle.

1.1 Preliminaries and Notation

Throughout this book we will denote by $(\Omega, \mathcal{F}, \mathrm{P})$ a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and P is a σ-additive probability measure on $(\Omega, \mathcal{F})$. If X is an integrable or nonnegative random variable on $(\Omega, \mathcal{F}, \mathrm{P})$, we denote by $\mathrm{E}(X)$ its expectation. For any $p \geq 1$, we denote by $L^p(\Omega)$ the space of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$ such that the norm

$$\|X\|_p := (\mathrm{E}(|X|^p))^{1/p}$$

is finite.

For any integers $k, n \geq 1$ we denote by $C_b^k(\mathbb{R}^n)$ the space of k-times continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$, such that f and all its partial derivatives of order up to k are bounded. We also denote by $C_0^k(\mathbb{R}^n)$ the subspace of functions in $C_b^k(\mathbb{R}^n)$ that have compact support. Moreover, $C_p^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^n$ that have at most polynomial growth together with their partial derivatives, $C_b^\infty(\mathbb{R}^n)$ is the subspace of functions in $C_p^\infty(\mathbb{R}^n)$ that are bounded together with their partial derivatives, and $C_0^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions with compact support.

1.2 Definition and Basic Properties

Brownian motion was named by Einstein (1905) after the botanist Robert Brown (1828), who observed in a microscope the complex and erratic mo-

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

tion of grains of pollen suspended in water. Brownian motion was then rigorously defined and studied by Wiener (1923); this is why it is also called the Wiener process. For extended expositions about Brownian motion see Revuz and Yor (1999), Mörters and Peres (2010), Durrett (2010), Bass (2011), and Baudoin (2014).

The mathematical definition of Brownian motion is the following.

Definition 1.2.1 A real-valued stochastic process $B = (B_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is called a *Brownian motion* if it satisfies the following conditions:

(i) Almost surely $B_0 = 0$.
(ii) For all $0 \leq t_1 < \cdots < t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1}$ are independent random variables.
(iii) If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.
(iv) With probability one, the map $t \to B_t$ is continuous.

More generally, a d-dimensional Brownian motion is defined as an $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t\geq 0}$, $B_t = (B_t^1, \dots, B_t^d)$, where $B^1, \dots, B^d$ are d independent Brownian motions.

We will sometimes consider a Brownian motion on a finite time interval $[0, T]$, which is defined in the same way.

Proposition 1.2.2 *Properties* (i), (ii), *and* (iii) *are equivalent to saying that B is a Gaussian process with mean zero and covariance function*

$$\Gamma(s, t) = \min(s, t). \tag{1.1}$$

Proof Suppose that (i), (ii), and (iii) hold. The probability distribution of the random vector $(B_{t_1}, \dots, B_{t_n})$, for $0 < t_1 < \cdots < t_n$, is normal because this vector is a linear transformation of the vector

$$\left(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}\right),$$

which has a normal distribution because its components are independent and normal. The mean $m(t)$ and the covariance function $\Gamma(s, t)$ are given by

$$\begin{aligned} m(t) &= \mathrm{E}(B_t) = 0, \\ \Gamma(s, t) &= \mathrm{E}(B_s B_t) = \mathrm{E}(B_s(B_t - B_s + B_s)) \\ &= \mathrm{E}(B_s(B_t - B_s)) + \mathrm{E}(B_s^2) = s = \min(s, t), \end{aligned}$$

if $s \leq t$. The converse is also easy to show. □

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

The existence of Brownian motion can be proved in different ways.

(1) The function $\Gamma(s,t) = \min(s,t)$ is symmetric and nonnegative definite because it can be written as

$$\min(s,t) = \int_0^\infty \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,t]}(r)dr.$$

Then, for any integer $n \geq 1$ and real numbers $a_1, \ldots, a_n$,

$$\begin{aligned}\sum_{i,j=1}^n a_i a_j \min(t_i, t_j) &= \sum_{i,j=1}^n a_i a_j \int_0^\infty \mathbf{1}_{[0,t_i]}(r)\mathbf{1}_{[0,t_j]}(r)dr \\ &= \int_0^\infty \left(\sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}(r)\right)^2 dr \geq 0.\end{aligned}$$

Therefore, by Kolmogorov's extension theorem (Theorem A.1.1), there exists a Gaussian process with mean zero and covariance function $\min(s,t)$.

Moreover, for any $s \leq t$, the increment $B_t - B_s$ has the normal distribution $N(0, t-s)$. This implies that for any natural number k we have

$$\mathrm{E}\left((B_t - B_s)^{2k}\right) = \frac{(2k)!}{2^k k!}(t-s)^k.$$

Therefore, by Kolmogorov's continuity theorem (Theorem A.4.1), there exists a version of B with Hölder-continuous trajectories of order γ for any $\gamma < (k-1)/(2k)$ on any interval $[0,T]$. This implies that the paths of this version of the process B are γ-Hölder continuous on $[0,T]$ for any $\gamma < 1/2$ and $T > 0$.

(2) Brownian motion can also be constructed as a Fourier series with random coefficients. Fix $T > 0$ and suppose that $(e_n)_{n\geq 0}$ is an orthonormal basis of the Hilbert space $L^2([0,T])$. Suppose that $(Z_n)_{n\geq 0}$ are independent random variables with law $N(0,1)$. Then, the random series

$$\sum_{n=0}^\infty Z_n \int_0^t e_n(r)dr \tag{1.2}$$

converges in $L^2(\Omega)$ to a mean-zero Gaussian process $B = (B_t)_{t\in[0,T]}$ with

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

covariance function (1.1). In fact, for any $s, t \in [0, T]$,

$$
\begin{aligned}
\mathrm{E}\Big(\Big(\sum_{n=0}^{N} Z_n \int_0^t e_n(r)dr\Big)\Big(\sum_{n=0}^{N} Z_n \int_0^s e_n(r)dr\Big)\Big)
&= \sum_{n=0}^{N}\Big(\int_0^t e_n(r)dr\Big)\Big(\int_0^s e_n(r)dr\Big) \\
&= \sum_{n=0}^{N} \langle \mathbf{1}_{[0,t]}, e_n\rangle_{L^2([0,T])} \langle \mathbf{1}_{[0,s]}, e_n\rangle_{L^2([0,T])},
\end{aligned}
$$

which converges as $N \to \infty$ to

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{L^2([0,T])} = \min(s, t).$$

The convergence of the series (1.2) is uniform in $[0, T]$ almost surely; that is, as N tends to infinity,

$$\sup_{0\le t\le T}\left|\sum_{n=0}^{N} Z_n \int_0^t e_n(r)dr - B_t\right| \xrightarrow{\text{a.s.}} 0. \tag{1.3}$$

The fact that the process B has continuous trajectories almost surely is a consequence of (1.3). We refer to Itô and Nisio (1968) for a proof of (1.3).

Once we have constructed the Brownian motion on an interval $[0, T]$, we can build a Brownian motion on $\mathbb{R}_+$ by considering a sequence of independent Brownian motions $B^{(n)}$ on $[0, T]$, $n \ge 1$, and setting

$$B_t = B_T^{(n-1)} + B_{t-(n-1)T}^{(n)}, \quad (n-1)T \le t \le nT,$$

with the convention $B_T^{(0)} = 0$.

In particular, if we take a basis formed by the trigonometric functions, $e_n(t) = (1/\sqrt{\pi})\cos(nt/2)$ for $n \ge 1$ and $e_0(t) = 1/\sqrt{2\pi}$, on the interval $[0, 2\pi]$, we obtain the Paley–Wiener representation of Brownian motion:

$$B_t = Z_0 \frac{t}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} Z_n \frac{\sin(nt/2)}{n}, \quad t \in [0, 2\pi]. \tag{1.4}$$

The proof of the construction of Brownian motion in this particular case can be found in Bass (2011, Theorem 6.1).

(3) Brownian motion can also be regarded as the limit in distribution of a symmetric random walk. Indeed, fix a time interval $[0, T]$. Consider n independent and identically distributed random variables $\xi_1, \dots, \xi_n$ with mean zero and variance T/n. Define the partial sums

$$R_k = \xi_1 + \cdots + \xi_k, \quad k = 1, \dots, n.$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

By the central limit theorem the sequence R_n converges in distribution, as n tends to infinity, to the normal distribution $N(0, T)$.

Consider the continuous stochastic process $S_n(t)$ defined by linear interpolation from the values

$$S_n\left(\frac{kT}{n}\right) = R_k, \quad k = 0, \dots, n.$$

Then, a functional version of the central limit theorem, known as the Donsker invariance principle, says that the sequence of stochastic processes $S_n(t)$ converges in law to Brownian motion on $[0, T]$. This means that, for any continuous and bounded function $\varphi\colon C([0, T]) \to \mathbb{R}$, we have

$$\mathrm{E}(\varphi(S_n)) \to \mathrm{E}(\varphi(B)),$$

as n tends to infinity.

Basic properties of Brownian motion are (see Exercises 1.5–1.8):

1. *Self-similarity* For any $a > 0$, the process $(a^{-1/2} B_{at})_{t\geq 0}$ is a Brownian motion.
2. For any $h > 0$, the process $(B_{t+h} - B_h)_{t\geq 0}$ is a Brownian motion.
3. The process $(-B_t)_{t\geq 0}$ is a Brownian motion.
4. Almost surely $\lim_{t\to\infty} B_t/t = 0$, and the process

$$X_t = \begin{cases} tB_{1/t} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

is a Brownian motion.

Remark 1.2.3 As we have seen, the trajectories of Brownian motion on an interval $[0, T]$ are Hölder continuous of order γ for any $\gamma < \frac{1}{2}$. However, the trajectories are not Hölder continuous of order $\frac{1}{2}$. More precisely, the following property holds (see Exercise 1.9):

$$\mathrm{P}\left(\sup_{s,t\in[0,1]} \frac{|B_t - B_s|}{\sqrt{|t-s|}} = +\infty \right) = 1.$$

The *exact modulus of continuity* of Brownian motion was obtained by Lévy (1937):

$$\limsup_{\delta\downarrow 0} \sup_{s,t\in[0,1],|t-s|<\delta} \frac{|B_t - B_s|}{\sqrt{2|t-s| \log|t-s|}} = 1, \quad \text{a.s.}$$

Lévy's proof can be found in Mörters and Peres (2010, Theorem 1.14). In

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

contrast, the behavior at a single point is given by the *law of the iterated logarithm*, due to Khinchin (1933):

$$\limsup_{t\downarrow s} \frac{|B_t - B_s|}{\sqrt{2|t-s|\log\log|t-s|}} = 1, \quad \text{a.s.}$$

for any $s \geq 0$. See also Mörters and Peres (2010, Corollary 5.3) and Bass (2011, Theorem 7.2).

Brownian motion satisfies $\mathrm{E}(|B_t - B_s|^2) = t - s$ for all $s \leq t$. This means that when $t - s$ is small, $B_t - B_s$ is of order $\sqrt{t-s}$ and $(B_t - B_s)^2$ is of order $t - s$. Moreover, the quadratic variation of a Brownian motion on $[0, t]$ equals t in $L^2(\Omega)$, as is proved in the following proposition.

Proposition 1.2.4 *Fix a time interval* $[0, t]$ *and consider the following subdivision* π *of this interval:*

$$0 = t_0 < t_1 < \cdots < t_n = t.$$

The norm of the subdivision π *is defined as* $|\pi| = \max_{0\leq j\leq n-1}(t_{j+1} - t_j)$. *The following convergence holds in* $L^2(\Omega)$*:*

$$\lim_{|\pi|\to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t. \tag{1.5}$$

Proof Set $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$. The random variables ξ_j are independent and centered. Thus,

$$\begin{aligned}
\mathrm{E}\left(\left(\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 - t\right)^2\right) &= \mathrm{E}\left(\left(\sum_{j=0}^{n-1}\xi_j\right)^2\right) = \sum_{j=0}^{n-1}\mathrm{E}\left(\xi_j^2\right) \\
&= \sum_{j=0}^{n-1}\left(3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2\right) \\
&= 2\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \leq 2t|\pi| \xrightarrow{|\pi|\to 0} 0,
\end{aligned}$$

which proves the result. □

As a consequence, we have the following result.

Proposition 1.2.5 *The total variation of Brownian motion on an interval* $[0, t]$*, defined by*

$$V = \sup_{\pi} \sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}|,$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

where $\pi = \{0 = t_0 < t_1 < \cdots < t_n\}$, is infinite with probability one.

Proof Using the continuity of the trajectories of Brownian motion, we have

$$\sum_{j=1}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 \leq \sup_j |B_{t_{j+1}} - B_{t_j}| \left(\sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}| \right)$$
$$\leq V \sup_j |B_{t_{j+1}} - B_{t_j}| \xrightarrow{|\pi|\to 0} 0$$

if $V < \infty$, which contradicts the fact that $\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2$ converges in mean square to t as $|\pi| \to 0$. Therefore, $\mathrm{P}(V < \infty) = 0$. □

Finally, the trajectories of B are almost surely nowhere differentiable. The first proof of this fact is due to Paley *et al.* (1933). Another proof, by Dvoretzky *et al.* (1961), is given in Durrett (2010, Theorem 8.1.6) and Mörters and Peres (2010, Theorem 1.27).

1.3 Wiener Integral

We next define the integral of square integrable functions with respect to Brownian motion, known as the Wiener integral.

We consider the set $\mathcal{E}_0$ of step functions

$$\varphi_t = \sum_{j=0}^{n-1} a_j \mathbf{1}_{(t_j,t_{j+1}]}(t), \quad t \geq 0, \tag{1.6}$$

where $n \geq 1$ is an integer, $a_0, \ldots, a_{n-1} \in \mathbb{R}$, and $0 = t_0 < \cdots < t_n$. The Wiener integral of a step function $\varphi \in \mathcal{E}_0$ of the form (1.6) is defined by

$$\int_0^\infty \varphi_t dB_t = \sum_{j=0}^{n-1} a_j (B_{t_{j+1}} - B_{t_j}).$$

The mapping $\varphi \to \int_0^\infty \varphi_t dB_t$ from $\mathcal{E}_0 \subset L^2(\mathbb{R}_+)$ to $L^2(\Omega)$ is linear and isometric:

$$\mathrm{E}\left(\left(\int_0^\infty \varphi_t dB_t\right)^2\right) = \sum_{j=0}^{n-1} a_j^2 (t_{j+1} - t_j) = \int_0^\infty \varphi_t^2 dt = \|\varphi\|^2_{L^2(\mathbb{R}_+)}.$$

The space $\mathcal{E}_0$ is a dense subspace of $L^2(\mathbb{R}_+)$. Therefore, the mapping

$$\varphi \to \int_0^\infty \varphi_t dB_t$$

© in this web service Cambridge University Press www.cambridge.org

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

can be extended to a linear isometry between $L^2(\mathbb{R}_+)$ and the Gaussian subspace of $L^2(\Omega)$ spanned by the Brownian motion. The random variable $\int_0^\infty \varphi_t dB_t$ is called the Wiener integral of $\varphi \in L^2(\mathbb{R}_+)$ and is denoted by $B(\varphi)$. Observe that it is a Gaussian random variable with mean zero and variance $\|\varphi\|^2_{L^2(\mathbb{R}_+)}$.

The Wiener integral allows us to view Brownian motion as the cumulative function of a white noise.

Definition 1.3.1 Let D be a Borel subset of $\mathbb{R}^m$. A *white noise* on D is a centered Gaussian family of random variables

$$\{W(A), A \in \mathcal{B}(\mathbb{R}^m), A \subset D, \ell(A) < \infty\},$$

where ℓ denotes the Lebesgue measure, such that

$$\mathrm{E}(W(A)W(B)) = \ell(A \cap B).$$

The mapping $\mathbf{1}_A \to W(A)$ can be extended to a linear isometry from $L^2(D)$ to the Gaussian space spanned by W, denoted by

$$\varphi \to \int_D \varphi(x) W(dx).$$

The Brownian motion B defines a white noise on $\mathbb{R}_+$ by setting

$$W(A) = \int_0^\infty \mathbf{1}_A(t) dB_t, \quad A \in \mathcal{B}(\mathbb{R}_+),\ \ell(A) < \infty.$$

Conversely, Brownian motion can be defined from white noise. In fact, if W is a white noise on $\mathbb{R}_+$, the process

$$W_t = W([0, t]), \quad t \geq 0,$$

is a Brownian motion.

The two-parameter extension of Brownian motion is the *Brownian sheet*, which is defined as a real-valued two-parameter Gaussian process $(B_t)_{t \in \mathbb{R}^2_+}$ with mean zero and covariance function

$$\Gamma(s, t) = \mathrm{E}(B_s B_t) = \min(s_1, t_1) \min(s_2, t_2), \quad s, t \in \mathbb{R}^2_+.$$

As above, the Brownian sheet can be obtained from white noise. In fact, if W is a white noise on $\mathbb{R}^2_+$, the process

$$W_t = W([0, t_1] \times [0, t_2]), \quad t \in \mathbb{R}^2_+,$$

is a Brownian sheet.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

1.4 Wiener Space

Brownian motion can be defined in the canonical probability space $(\Omega, \mathcal{F}, \mathrm{P})$ known as the Wiener space. More precisely:

- Ω is the space of continuous functions $\omega\colon \mathbb{R}_+ \to \mathbb{R}$ vanishing at the origin.
- $\mathcal{F}$ is the Borel σ-field $\mathcal{B}(\Omega)$ for the topology corresponding to uniform convergence on compact sets. One can easily show (see Exercise 1.11) that $\mathcal{F}$ coincides with the σ-field generated by the collection of cylinder sets

$$C = \{\omega \in \Omega : \omega(t_1) \in A_1, \dots, \omega(t_k) \in A_k\}, \tag{1.7}$$

  for any integer $k \geq 1$, Borel sets $A_1, \dots, A_k$ in $\mathbb{R}$, and $0 \leq t_1 < \cdots < t_k$.
- P is the Wiener measure. That is, P is defined on a cylinder set of the form (1.7) by

$$\mathrm{P}(C) = \int_{A_1 \times \cdots \times A_k} p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \cdots p_{t_k - t_{k-1}}(x_k - x_{k-1})\, dx_1 \cdots dx_k, \tag{1.8}$$

  where $p_t(x)$ denotes the Gaussian density

$$p_t(x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}, \qquad x \in \mathbb{R}, t > 0.$$

The mapping P defined by (1.8) on cylinder sets can be uniquely extended to a probability measure on $\mathcal{F}$. This fact can be proved as a consequence of the existence of Brownian motion on $\mathbb{R}_+$. Finally, the canonical stochastic process defined as $B_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t \geq 0$, is a Brownian motion.

The canonical probability space $(\Omega, \mathcal{F}, \mathrm{P})$ of a d-dimensional Brownian motion can be defined in a similar way.

Further into the text, $(\Omega, \mathcal{F}, \mathrm{P})$ will denote a general probability space, and only in some special cases will we restrict our study to Wiener space.

1.5 Brownian Filtration

Consider a Brownian motion $B = (B_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. For any time $t \geq 0$, we define the σ-field $\mathcal{F}_t$ generated by the random variables $(B_s)_{0 \leq s \leq t}$ and the events in $\mathcal{F}$ of probability zero. That is, $\mathcal{F}_t$ is the smallest σ-field that contains the sets of the form

$$\{B_s \in A\} \cup N,$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-03912-4 — Introduction to Malliavin Calculus
David Nualart , Eulalia Nualart
Excerpt
More Information

where $0 \le s \le t$, A is a Borel subset of $\mathbb{R}$, and $N \in \mathcal{F}$ is such that $\mathrm{P}(N) = 0$. Notice that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \le t$; that is, $(\mathcal{F}_t)_{t\ge 0}$ is a nondecreasing family of σ-fields. We say that $(\mathcal{F}_t)_{t\ge 0}$ is the *natural filtration* of Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Inclusion of the events of probability zero in each σ-field $\mathcal{F}_t$ has the following important consequences:

1. Any version of an adapted process is also adapted.
2. The family of σ-fields is right-continuous; that is, for all $t \ge 0$, $\cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$.

Property 2 is a consequence of Blumenthal's 0–1 law (see Durrett, 2010, Theorem 8.2.3).

The natural filtration $(\mathcal{F}_t)_{t\ge 0}$ of a d-dimensional Brownian motion can be defined in a similar way.

1.6 Markov Property

Consider a Brownian motion $B = (B_t)_{t\ge 0}$. The next theorem shows that Brownian motion is an $\mathcal{F}_t$-*Markov process* with respect to its natural filtration $(\mathcal{F}_t)_{t\ge 0}$ (see Definition A.5.1).

Theorem 1.6.1 *For any measurable and bounded (or nonnegative) function $f : \mathbb{R} \to \mathbb{R}$, $s \ge 0$ and $t > 0$, we have*

$$\mathrm{E}(f(B_{s+t})|\mathcal{F}_s) = (P_t f)(B_s),$$

where

$$(P_t f)(x) = \int_{\mathbb{R}} f(y) p_t(x-y) dy.$$

Proof We have

$$\mathrm{E}(f(B_{s+t})|\mathcal{F}_s) = \mathrm{E}(f(B_{s+t} - B_s + B_s)|\mathcal{F}_s).$$

Since $B_{s+t} - B_s$ is independent of $\mathcal{F}_s$, we obtain

$$\begin{aligned}\mathrm{E}(f(B_{s+t})|\mathcal{F}_s) &= \mathrm{E}(f(B_{s+t} - B_s + x))|_{x=B_s} \\ &= \int_{\mathbb{R}} f(y + B_s)\frac{1}{\sqrt{2\pi t}} e^{-|y|^2/(2t)} dy \\ &= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}} e^{-|B_s - y|^2/(2t)} dy = (P_t f)(B_s),\end{aligned}$$

which concludes the proof. □

© in this web service Cambridge University Press www.cambridge.org