Cambridge University Press & Assessment
978-1-107-03769-4 — Problems for Biomedical Fluid Mechanics and Transport Phenomena
Mark Johnson , C. Ross Ethier
Excerpt
More Information

1 Problem solving

In this introductory chapter, we begin with a derivation of the Reynolds transport theorem, which is central to conservation principles applied to control volumes. Then, we turn to the issue of how to approach problem solving.

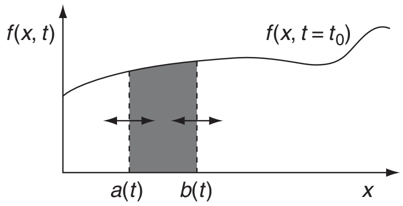
1.1 The Reynolds transport theorem

Quantities, such as mass, momentum, energy and even entropy and money, are conserved in the sense that the following principle can be applied to a system.

Input + Generation = Output + Accumulation

The system normally considered in transport phenomena for application of this principle is a control volume. The equation makes intuitive sense and is simple to apply in many cases. However, when moving control volumes and reference frames are examined, or when transport of quantities that have direction (such as momentum) is considered, intuition is less reliable. We here derive a rigorous version of this conservation principle and, in the process, discover the wide applicability of the Reynolds transport theorem. We note that more intuitive formulations of this principle can be found in other texts (e.g. *Fluid Mechanics* by Potter and Foss).

We consider a generalization of Leibniz's rule for the differentiation of integrals. Consider a given function[1] $f(x)$ and the definite integral (M) of this function between $x = a$ and $x = b$. Let both this function and the limits of integration be functions of time (t) (see the figure):



[1] Note that f could be either a scalar- or a vector-valued function. We write it here as a scalar (unbolded).

$$M=\int_{a(t)}^{b(t)} f(x,t)dx$$

Using the chain rule, we can find how the value of this integral changes with time:

$$\frac{dM}{dt}=\frac{\partial M}{\partial t}+\frac{\partial M}{\partial a}\frac{da}{dt}+\frac{\partial M}{\partial b}\frac{db}{dt}$$

or

$$\frac{dM}{dt}=\int_{a(t)}^{b(t)}\frac{\partial f(x,t)}{\partial t}dt+f[b(t),t]\frac{db}{dt}-f[a(t),t]\frac{da}{dt}$$

This is Leibniz's rule, which is well known from calculus. M changes with time not only due to temporal changes in f, but also because the boundaries of integration move. Note that the temporal derivative was taken inside the integral, since a and b are held constant in the partial derivative. This will be important when we consider moving control volumes.

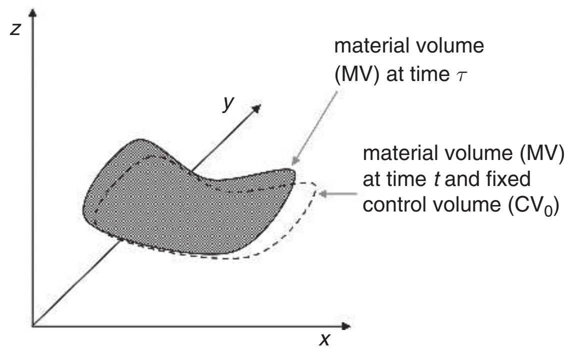
We now look to apply a similar principle but in three dimensions, relating the time rate of change of a moving system to that of a stationary system. This is particularly important in transport phenomena, not simply because our systems are frequently moving, but, more importantly, because our laws of physics are derived for material volumes, not control volumes.

A material volume is a fixed, identifiable set of matter.[2] A control volume is a region of space, fixed or moving, that we choose to analyze. Our laws of physics apply directly to matter, not to control volumes. For example, physics tells us that (for non-relativistic systems) mass is conserved. Thus, the mass of a given material volume is always constant. But the mass in a control volume can change.

Solving a problem by tracking the moving material volume is known as a Langrangian approach. It is typically quite difficult to solve problems in this way since material volumes change their location and shape due to their motion. Analysis is facilitated by use of a control volume whose shape and motion can be specified; such an approach is known as Eulerian. However, to use an Eulerian approach, we require the Reynolds transport theorem, which allows us to relate physical laws that are derived for material volumes to a principle that applies to

[2] Also referred to by some authors as a "control mass." Note that the use of the term "fixed" in the above definition does not imply that the material volume is not moving; rather, it means that its constituent parts are neither destroyed nor created, although they can be transformed into other components through e.g. chemical reactions.

control volumes. In other words, the Reynolds transport theorem acts as a "bridge" between material volumes, where the physical laws are defined, and control volumes, which are more convenient for analysis.



Consider a moving material volume as shown in the figure above. This material volume is moving such that it occupies the region surrounded by the dashed line at time t and the solid line at a later time τ. Note that the points within the material volume are not all necessarily moving with the same velocity (e.g. a fluid or a deforming solid).

Pick a control volume that coincides with the material volume at time t. We define M as the integral of a function $f(\boldsymbol{x}, t)$ over the material volume,

$$M = \int_{\mathrm{MV}} f(\boldsymbol{x}, t) d\boldsymbol{x}$$

where $\boldsymbol{x} = (x, y, z)$. We will relate M to the integral of the same function, $f(\boldsymbol{x}, t)$, over the control volume.

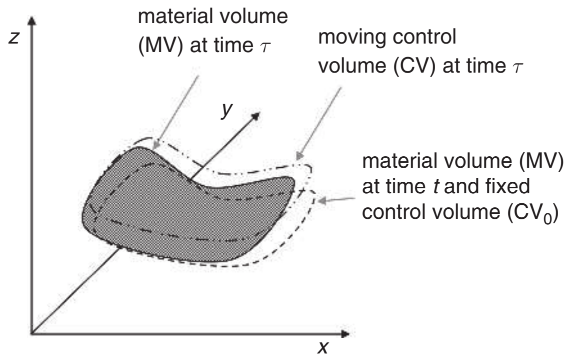
We use an analogous approach to that leading to Leibniz's equation. We consider the integral of a function $f(\boldsymbol{x}, t)$ over the material volume. M changes with time due both to temporal changes in $f(\boldsymbol{x}, t)$ and to the motion of the boundary of the domain of integration. Noting that the final two terms in Leibniz's equation arise due to the flux of f at the boundary carried by the material's velocity out of the control volume (and thus normal to the control surface), we find that the three-dimensional equivalent of Leibniz's equation becomes

$$\frac{dM}{dt} = \int_{\mathrm{CV}_0} \frac{\partial f(\boldsymbol{x}, t)}{\partial t} d\boldsymbol{x} + \int_{\mathrm{CS}_0} f(\boldsymbol{x}, t) \left(\vec{V}_{\mathrm{MV}} \cdot \hat{n} \right) dS$$

where CS_0 is the surface surrounding the control volume CV_0, $\vec{V}_{MV}$ is the velocity of the material volume, and $\hat{n}$ is the outward pointing unit normal.

This is the Reynolds transport theorem for a stationary control volume. It relates the time rate of change of an intensive function, f (a parameter per unit volume), integrated over a material volume to the integral of that intensive function integrated over a control volume. The second integral over the control surface involves the flux of material entering or leaving the control volume. Note that no material enters or leaves a material volume (by definition).

It is frequently convenient when solving transport problems to consider moving control volumes. To generalize the Reynolds transport theorem, consider both a stationary control volume CV_0 and a control volume CV moving at velocity $\vec{V}_{CV}$, and their respective surfaces, CS_0 and CS (see the figure below).



Now, to find the generalized Reynolds transport theorem for the moving control volume, we use the Reynolds transport theorem twice: the first time relating the moving material volume to the stationary control volume, and the second time relating the moving control volume to the stationary control volume:

$$\frac{d}{dt}\int_{MV} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{CV_0} \frac{\partial f(\boldsymbol{x},t)}{\partial t} d\boldsymbol{x} + \int_{CS_0} f(\boldsymbol{x},t)\left(\vec{V}_{MV}\cdot\hat{n}\right)dS$$

$$\frac{d}{dt}\int_{CV} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{CV_0} \frac{\partial f(\boldsymbol{x},t)}{\partial t} d\boldsymbol{x} + \int_{CS_0} f(\boldsymbol{x},t)\left(\vec{V}_{CV}\cdot\hat{n}\right)dS$$

On subtracting the second equation from the first, rearranging, and evaluating at time t when the two control volumes are coincident (so that CS and CS_0 are identical), we find

$$\frac{d}{dt}\int_{\mathrm{MV}} f(\boldsymbol{x},t)d\boldsymbol{x} = \frac{d}{dt}\int_{\mathrm{CV}} f(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} f(\boldsymbol{x},t)\left(\overrightarrow{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

where $\overrightarrow{V}_{\mathrm{rel}}$ is the velocity of the material volume relative to the moving control volume. This is the general form of the Reynolds transport theorem, and it is valid for stationary and moving control volumes.

The physical interpretation of this equation is useful. This is a conservation law for any conserved quantity *f*, in which *f* is an intensive variable (expressed per unit volume). The term on the left-hand side of the equation is the rate at which *f* is generated. The first term on the right-hand side of the equation is the accumulation term: the rate at which *f* accumulates in the control volume. The final term is the flux term, characterizing the balance of the flux of *f* out of and into the control volume due to flow. Thus, the Reynolds transport theorem recovers our initial conservation principle, namely

Generation = Accumulation + Output – Input

1.2 Application of the Reynolds transport theorem

By applying the laws of physics to the left-hand side of this equation, conservation laws that apply to control volumes can be generated. For example, when considering mass conservation, the function *f* becomes the fluid density ρ (mass per unit volume, an intensive variable). Then the left-hand side of the equation is simply the time rate of change of the mass of the material volume. Since this mass is constant (mass is not generated), we find that

$$0 = \frac{d}{dt}\int_{\mathrm{CV}} \rho(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} \rho(\boldsymbol{x},t)\left(\overrightarrow{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

This is the mass-conservation equation, which is valid for all non-relativistic control volumes, indicating that accumulation in a control volume results from an imbalance between the influx and outflow of mass from a control volume.

For species conservation, we let $f = C_i$ (moles of species *i* per unit volume). There are two important differences from the law of mass conservation. First, there is the possibility of generation or destruction of species *i* due to chemical reactions. We will let the net generation rate of species *i* be Ψ_i, i.e. the production rate minus the destruction rate. Second, in addition to the flow carrying species *i* ($C_i\overrightarrow{V}_{\mathrm{rel}}$), the diffusion of this species needs to be accounted for.

The diffusional flux of species *i* is given by Fick's law of diffusion[3]: $\overrightarrow{j_i} = -D_i\nabla C_i$, where D_i is the diffusion coefficient of species *i*. Taking the dot

[3] For isothermal, isobaric conditions.

product of this vector with the unit outward normal to the control surface and integrating over the control surface gives the total net diffusional transport out of the control volume. We then use the Reynolds transport theorem to find the species conservation equation:

$$\int_{\mathrm{CV}} \Psi_i(\boldsymbol{x},t)d\boldsymbol{x} = \frac{d}{dt}\int_{\mathrm{CV}} C_i(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} C_i(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS + \int_{\mathrm{CS}} (\vec{J}_i(\boldsymbol{x},t)\cdot\hat{n})dS$$

Likewise, if we allow that $f = \rho\vec{V}$ (momentum per unit volume, a vector), then the left-hand side of the Reynolds transport theorem is the time rate of change of the momentum of the material volume. This we know from Newton's second law must be the sum of the forces acting on the material volume. Thus, the momentum equation is derived:

$$\sum \vec{F} = \frac{d}{dt}\int_{\mathrm{CV}} \rho\vec{V}(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} \rho\vec{V}(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

This is a vector equation that describes a momentum balance in each of the coordinate directions.

Note that we have imposed no restrictions on the motion of our control volume when deriving the momentum-conservation equation. It can even be accelerating. However, the reference frame (which is not the same as the control volume) cannot be accelerating because Newton's second law does not hold (without modification) for non-inertial reference frames.

Note also that the second integral in the above equation contains two velocities that are not necessarily the same. One is the velocity of the material volume (the fluid), while the other is the relative velocity between the fluid and the control volume. The velocities can even be in different directions (e.g. transferring x-momentum in the y-direction such as might occur when one skater passes another and throws a book in a perpendicular direction that is caught by the slower skater).

The relative velocity in the last term of the momentum equation is present as the dot product with the outward normal, so only the component of $\vec{V}_{\mathrm{rel}}$ that carries material across the control surface contributes to the integral. The sign of a term can be confusing to determine: the sign of any component of $\vec{V}$ is established by the coordinate direction, e.g. a positive V_x is one that points in the same direction as the x-axis. However, the sign of the term $\vec{V}_{\mathrm{rel}}\cdot\hat{n}$ is determined only by whether fluid is entering or leaving the control volume, being negative or positive, respectively. Students must pay attention to this tricky point!

Application of the Reynolds transport theorem for other parameters can yield equations of angular-momentum conservation or energy conservation. Some of the problems in this book are best tackled using the conservation principles applied to judiciously chosen control volumes.

Unfortunately, many errors are made in applying these principles. Students frequently have difficulties applying the various forms of the Reynolds transport theorem to moving control volumes, especially when vector quantities are involved. The relative velocity that appears in the Reynolds transport theorem is frequently a source of confusion, as noted above.

Students generally have difficulty in approaching transport problems. This is in no small part due to difficulty in thinking in terms of an Eulerian analysis, since much of the physics that students first learn is necessarily taught from a Lagrangian point of view. We now turn to the more general topic of problem solving.

1.3 Approaching transport problems

It is our experience that many students have difficulty with problem solving because they start "in the middle," i.e. write down a conservation law and begin to use it without first deciding **what** they want to conserve and how they should approach the problem. As is the case in so many things in life, it pays to invest some effort in deciding how you want to tackle the problem before diving in. We provide here a set of steps designed to help you make this investment.

1. Draw a GOOD figure. Include all relevant aspects of the problem – the more detailed the figure, the better. Include the physical dimensions of the system and the physical properties of the materials which are given in the problem statement. Decide on sign conventions and a datum (if needed). For example, which direction will be positive and which negative? Where is the origin? (This will be required for use of linear- or angular-momentum conservation. Write the positive directions on your diagram.) Note that, once you have established a positive direction, you have to be consistent: you can't have the x-component of velocity be considered positive rightward but the x-component of force be positive leftward!

Use physical insights to help you characterize the process (e.g. draw streamlines on the figure). List your assumptions. Take some time to do step 1 well, because it is important!

2. Decide what physical law(s) and equation(s) you might want to apply.

- Mass conservation (and, in problems with more than one species, each obeys a conservation principle)
- Linear-momentum conservation and in which direction. Note that linear-momentum conservation cannot be applied in the radial direction (why?).
- Angular-momentum conservation
- Energy conservation
- Bernoulli's equation
- Fick's law of diffusion
- The Navier–Stokes equations
- The convection–diffusion equation

If unsure where to begin, start with application of the simplest law (mass conservation) and progress from there. Students often ask "how do I know what laws to use?" You will learn this by doing problems and gathering experience. Unfortunately, there is no short-cut that substitutes for experience in this case.

3. Pick an object to apply the physical law to. This is a crucial step. This can be a control volume, a free-body diagram, or a streamline (and, if one is applying a differential equation such as the Navier–Stokes equation, a domain must be chosen). Draw this on your figure (don't just visualize it in your head). In the case of a control volume, you **must** give careful consideration to where the boundaries of the volume will lie. You should pick these boundaries according to the "know or want to know" principle: the boundaries should lie either at locations at which your variables are specified or at locations at which you wish to determine the value of a variable. Also, for the application of momentum conservation, avoid cutting any objects with the control surface, since this introduces an unknown force or stress at this location (unless this is the desired result).

If you are going to use a streamline with Bernoulli's equation, you need to choose the location of the starting and ending points of the streamline and draw them onto your diagram. It is difficult to talk about the pressure, velocity, and elevation at point 1 without knowing exactly where point 1 is!

Note that judicious choice of the object you intend to apply a particular physical law to can make problems much easier to solve. However, there is often more than one choice that will lead you to the solution! Some choices might not give you the information you want, or may make it more difficult to solve the problem, but they are not actually wrong. (For example, you could end up proving that $x = x$, which is not useful but at least is not incorrect.) It is better to pick an object and get started, rather than to sit and debate what the best object or principle to use is.

Finally, regarding control volumes, it is necessary to explicitly decide whether the control volume is moving or stationary, and whether it retains the same shape or is deforming with time. Note that the control volume can even be accelerating, although the reference frame must be inertial (not accelerating) in order to allow use of the momentum theorem (unless an altered form of momentum conservation that introduces the inertial accelerations as pseudo-forces is used).

4. Apply equations to your objects. Note that different physical laws can be applied to different objects. Or, the same law can be applied to different objects. Note the total number of equations.

5. Determine the number of unknowns, remembering that some of your unknowns can be vectors that represent two or three unknown values (in two and three dimensions, respectively). Luckily, in such cases the governing equations are also vector-valued. If you have more unknowns than equations, go back to step 2 and pick another principle and another object to apply this principle to.

6. When the number of equations equals the number of unknowns, solve the equations. Identify any boundary conditions and initial conditions that may be needed to solve the equations, and indicate these on your diagram. Make sure that you have a sufficient number of these given the number of equations and the order of any differential equations.

7. When you obtain your solution, check to make sure that it satisfies your assumptions (e.g. that the Reynolds number is within the assumed range, your assumed streamline is a streamline, etc.).

8. Check the units of your answer. If the units are wrong, go backwards one step at a time to find where the units error occurred. This is an extremely important step that many students miss (regardless of how many times they are told). We refer those readers who believe that units are not important, or that they can be "filled in at the end," to the story of the Gimli Glider (see e.g. en.wikipedia.org/wiki/Gimli_Glider) or the Mars Climate Orbiter (en.wikipedia.org/wiki/Mars_Climate_Orbiter).

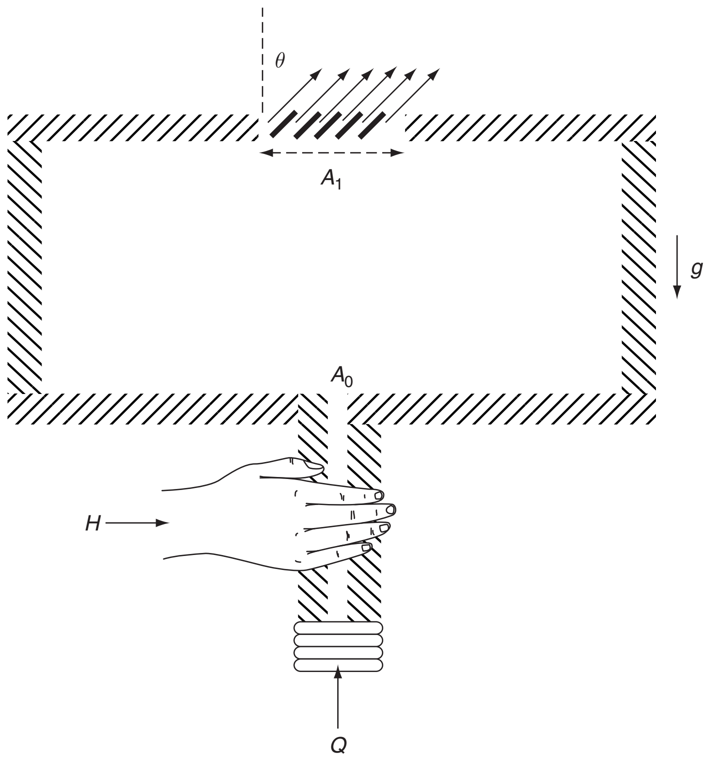
In this vein, it is also better to leave all equations in symbolic form and not to plug in numbers until the very last step. This not only makes unit checking easier, but also makes your work easier to follow.

9. Check to make sure your answer makes physical sense (e.g. motion in the correct direction, order of magnitude reasonable, boundary conditions satisfied, common-sense check). It's important not to skip this step! It provides closure to the problem, and allows you to understand in physical terms what your solution implies.

1.4 An example

Now, this all seems straightforward enough. Let's see how we use these principles to solve a problem.

Consider the figure shown overleaf: water (density ρ) enters the box from below through a flexible set of bellows at a flow rate Q, passes through the box and then, at



the exit, is directed to the right (at an angle θ) by the slots in the top of the box. We want to find the horizontal force (H) necessary to hold this system stationary under steady-state conditions. We suggest that, before proceeding, you try to work out this problem for yourself.

1. **Draw a good figure**. We begin by re-drawing the figure given in the problem description (do not skip this step), and adding a coordinate system to the figure along with the velocity of the fluid exiting the box. Note that, by observing that we are expressing the velocity of the fluid at the exit as a vector, we now realize that there are two unknowns here.

2. We now decide **what physical laws to apply**. It seems that, if we are to find the force H, then we must apply momentum conservation. However, it is always a good idea to start with mass conservation. This will constrain any answer that we may find, and may give additional insights into the problem.

3. **Pick a control volume**. We know the flow rate Q entering the system at the bottom, so it is useful to have a control surface at this location. Since it will be useful to know the velocity of the fluid leaving the system at the top, it will also be useful to have a control surface at the top of the chamber where the flow leaves the