1 Multivariable Calculus

We commence with a presentation of fundamental notions from the *calculus of many variables*, a subject that will be seen to provide a useful language to describe multidimensional geometry, a topic foundational to nearly all of engineering. We focus on presenting its grammar in sufficient richness to enable precise rendering of both simple and more complex geometries. This is of self-evident importance in engineering applications. We begin with a summary of implicitly defined functions and, in what will be a foundation for the book, local linearization. This quickly leads us to consider matrices that arise from linearization. The basic strategies of optimization, both unconstrained and constrained, are considered via identification of maxima and minima as well as the calculus of variations and Lagrange multipliers. The chapter ends with an extensive exposition relating multivariable calculus to the geometry of nonlinear coordinate transformations; in so doing, we develop geometry-based analytical tools that are used throughout the book. Such tools have relevance in fields ranging from computational mechanics widely used in engineering software to the theory of dynamical systems. Understanding the general nonlinear case also prepares one to better frame the more common geometrical methods associated with orthonormal coordinate transformations studied in Chapter 2.

1.1 Implicit Functions

1.1.1 One Independent Variable

We begin with a consideration of implicit functions of the form

$$f(x, y) = 0. \tag{1.1}$$

Here f is allowed to be a fully nonlinear function of x and y. We take the point (x_0, y_0) to satisfy $f(x_0, y_0) = 0$. Near (x_0, y_0), we can locally linearize via the familiar *total differential*,

$$df = \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} dx + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} dy = 0. \tag{1.2}$$

At the point (x_0, y_0), both $\partial f/\partial x$ and $\partial f/\partial y$ are constants, and we consider the local "variables" to be dx and dy, the infinitesimally small deviations of x and y from x_0

and y_0. As long as $\partial f/\partial y \neq 0$, we can scale to say

$$\left.\frac{dy}{dx}\right|_{x_0,y_0} = -\frac{\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}}{\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}}. \tag{1.3}$$

And with the local existence of dy/dx, we can consider y to be a unique local linear function of x and quantify its variation as

$$y - y_0 \approx \left.\frac{dy}{dx}\right|_{x_0,y_0} (x - x_0). \tag{1.4}$$

The process of examining the derivative at a point is a fundamental part of the use of linearization to gain local knowledge of a nonlinear entity. From here on we generally do not specify the point at which the evaluation is occurring, as it should be easy to understand from the context of the problem.

These notions can be formalized in the following:

Implicit Function Theorem: For a given $f(x, y)$ with $f = 0$ and $\partial f/\partial y \neq 0$ at the point (x_0, y_0), there corresponds a unique function $y(x)$ in the neighborhood[1] of (x_0, y_0).

When such a condition is satisfied, we can consider the dependent variable y to be locally a unique function of the independent variable x. The theorem says nothing about the case $\partial f/\partial y = 0$. Though such a case explicitly states that at such a point f is not a function of y, it does not speak to $y(x)$.

EXAMPLE 1.1

Consider the implicit function

$$f(x, y) = x^2 - y = 0. \tag{1.5}$$

Determine where y can be considered to be a unique local function of x.

We first recognize that when $f = 0$, we have $y = x^2$, which is a simple parabola. And it is easy to see that for all $x \in (-\infty, \infty)$, y is defined[2] and will be such that $y \in [0, \infty)$. Let us apply the formalism of the implicit function theorem to see how it is exercised on this well-understood curve. Applying Eq. (1.2) to our f, we find

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0, \tag{1.6}$$

$$= 2x\, dx - \, dy = 0. \tag{1.7}$$

Thus,

$$\frac{dy}{dx} = 2x. \tag{1.8}$$

This is defined for all $x \in (-\infty, \infty)$. Thus, a unique linearization $y(x)$ exists for all $x \in (-\infty, \infty)$. In the neighborhood of a point (x_0, y_0), we have the local linearization

$$y - y_0 = 2x_0(x - x_0). \tag{1.9}$$

[1] *Neighborhood* is a word whose mathematical meaning is close to its common meaning; nevertheless, it does require a precise definition, which is deferred until Section 6.1.

[2] Here $\in$ stands for "is an element of" and indicates that x could be any number, which we assume to be real. Also, we use the standard notation $(\cdot)$ to describe an open interval on the real axis, that is, one that does not include the endpoints. A closed interval is denoted by $[\cdot]$ and mixed intervals by either $(\cdot]$ or $[\cdot)$.

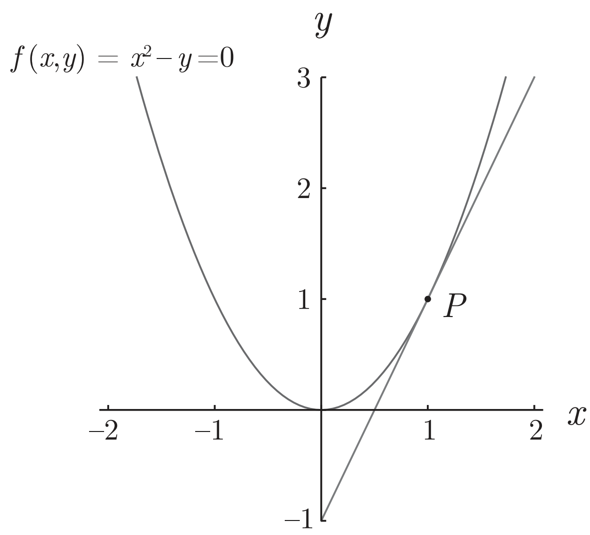

Figure 1.1. The parabola defined implicitly by $f(x, y) = x^2 - y = 0$ along with its local linearization near $P : (1, 1)$.

Consider the point $P : (x_0, y_0) = (1, 1)$. Obviously at P, $f = 1^2 - 1 = 0$, so P is on the curve. The local linearization at P is

$$y - 1 = 2(x - 1), \tag{1.10}$$

which shows uniquely how y varies with x in the neighborhood of P. The parabola defined by $f(x, y) = x^2 - y = 0$ and the linearization near P are plotted in Figure 1.1.

EXAMPLE 1.2

Consider the implicit function

$$f(x, y) = x^2 + y^2 - 1 = 0. \tag{1.11}$$

Determine where y can be considered to be a unique local function of x.

We first note that f is obviously nonlinear; our analysis will hinge on its local linearization. The function can be rewritten as $x^2 + y^2 = 1$, which is recognized as a circle centered at the origin with radius of unity. By inspection, we will need $x \in [-1, 1]$ to have a real value of y. Let us see how the formalism of the implicit function theorem applies. We have $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$. We are thus concerned that wherever $y = 0$, there is no unique function $y(x)$ that can be defined in the neighborhood of such a point. Applying Eq. (1.2) to our f, we find

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0, \tag{1.12}$$

$$= 2x\,dx + 2y\,dy = 0. \tag{1.13}$$

Thus,

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{1.14}$$

Obviously, whenever $y = 0$, the derivative dy/dx is undefined, and there is no local linearization. In the neighborhood of a point (x_0, y_0), with $y_0 \neq 0$, we have the local linearization

$$y - y_0 = -\frac{x_0}{y_0}(x - x_0). \tag{1.15}$$

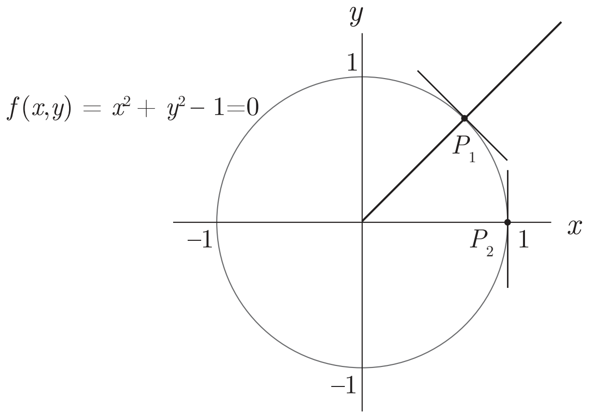

Figure 1.2. The circle defined implicitly by $f(x,y) = x^2 + y^2 - 1 = 0$ along with its local linearization near $P_1 : (\sqrt{2}/2, \sqrt{2}/2)$ and $P_2 : (1, 0)$.

Thus, while y exists for $x \in [-1, 1]$, we only have a local linearization for $x \in (-1, 1)$. When $x = \pm 1$, $y = 0$, and the local linearization does not exist.

Consider two points on the circle $P_1 : (x_0, y_0) = (\sqrt{2}/2, \sqrt{2}/2)$ and $P_2 : (x_0, y_0) = (1, 0)$. Obviously $f = 0$ at both P_1 and P_2, so both are on the circle. At P_1, we have the local linearization

$$y - \frac{\sqrt{2}}{2} = -\left(x - \frac{\sqrt{2}}{2}\right), \tag{1.16}$$

showing uniquely how y varies with x near P_1, the curve being the tangent line to f. At P_2, it is obvious that the tangent line is expressed by $x = 1$, which does not tell us how y varies with x locally. Indeed, we do know that $y = 0$ at P_2, but we lack a unique formula for its variation with x in nearby regions.

The circle defined by $f(x,y) = x^2 + y^2 - 1 = 0$ and the linearizations near P_1 and P_2 are plotted in Figure 1.2. Globally, of course, y does vary with x for this f; in fact, one can see by inspection that

$$y(x) = \pm\sqrt{1 - x^2}. \tag{1.17}$$

This function takes on real values for $x \in [-1, 1]$, but within that domain, it is nowhere unique. One might think a Taylor[3] series (see Section 5.1.1) approximation of Eq. (1.17) might be of use near P_2, but that too is nonunique, yielding the nonlinear approximations

$$y(x) \approx \sqrt{2(1 - x)}, \tag{1.18}$$

$$y(x) \approx -\sqrt{2(1 - x)}, \tag{1.19}$$

for $x \approx 1$. So the impact of the implicit function theorem on this f at $P_2 : (1, 0)$ is that there is no *unique* approximating linear function $y(x)$ at P_2. Whenever a linear approximation is available, we can expect uniqueness.

Had our global function been $f(x,y) = x - 1 = 0$, we would have had $\partial f/\partial y = 0$ everywhere and would nowhere have been able to identify $y(x)$; this simply reflects that for $f = x - 1 = 0$, whenever $x = 1$, any value of y will allow satisfaction of $f = 0$.

[3] Brook Taylor, 1685–1731, English mathematician, musician, and painter.

1.1.2 Many Independent Variables

If, instead, we have N independent variables, we can think of a relation such as

$$f(x_1, x_2, \ldots, x_N, y) = 0 \tag{1.20}$$

in some region as an implicit function of y with respect to the other variables. We cannot have $\partial f/\partial y = 0$, because then f would not depend on y in this region. In principle, we can write

$$y = y(x_1, x_2, \ldots, x_N) \tag{1.21}$$

if $\partial f/\partial y \neq 0$.

The derivative $\partial y/\partial x_n$ can be determined from $f = 0$ without explicitly solving for y. First, from the definition of the total differential, we have

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \frac{\partial f}{\partial x_2}\, dx_2 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n + \cdots + \frac{\partial f}{\partial x_N}\, dx_N + \frac{\partial f}{\partial y}\, dy = 0. \tag{1.22}$$

Differentiating with respect to x_n while holding all the other $x_m, m \neq n$, constant, we get

$$\frac{\partial f}{\partial x_n} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x_n} = 0, \tag{1.23}$$

so that

$$\frac{\partial y}{\partial x_n} = -\frac{\frac{\partial f}{\partial x_n}}{\frac{\partial f}{\partial y}}, \tag{1.24}$$

which can be found if $\partial f/\partial y \neq 0$. That is to say, y can be considered a function of x_n if $\partial f/\partial y \neq 0$. Again, local linearization of a nonlinear function provides a key to understanding the behavior of a function in a small neighborhood.

EXAMPLE 1.3

Consider the implicit function

$$f(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 - 1 = 0. \tag{1.25}$$

Determine where y can be considered to be a unique local function of x_1 and x_2.

This problem is essentially a three-dimensional extension of the previous example. Here the nonlinear $f = 0$ describes a sphere centered at the origin with radius unity: $x_1^2 + x_2^2 + y^2 = 1$. We have $\partial f/\partial x_1 = 2x_1$, $\partial f/\partial x_2 = 2x_2$, and $\partial f/\partial y = 2y$. We thus do not expect a unique $y(x_1, x_2)$ to describe the points on the sphere in the neighborhood of $y = 0$.

Applying Eq. (1.22) to our f, we find

$$df = 2x_1\, dx_1 + 2x_2\, dx_2 + 2y\, dy = 0. \tag{1.26}$$

Holding x_2 constant, we can say

$$\frac{\partial y}{\partial x_1} = -\frac{x_1}{y}. \tag{1.27}$$

Holding x_1 constant, we can say

$$\frac{\partial y}{\partial x_2} = -\frac{x_2}{y}. \tag{1.28}$$

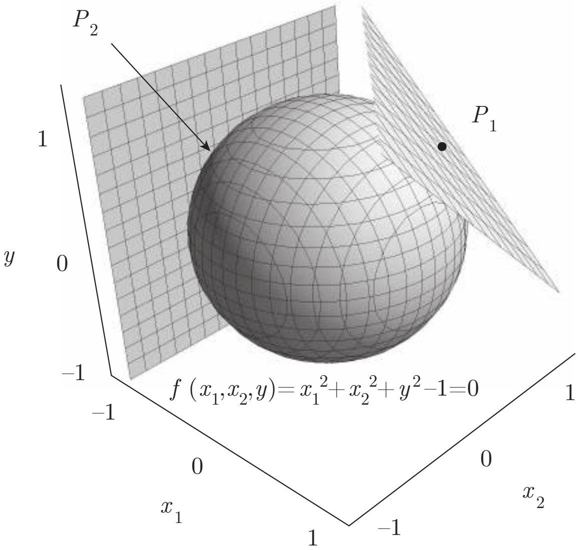

Figure 1.3. The sphere defined implicitly by $f(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 - 1 = 0$ along with its local tangent planes at P_1 : $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and P_2 : $(-1, 0, 0)$.

Near an arbitrary point (x_{1a}, x_{2a}, y_a) on the surface $f = 0$, we get the equation for the *tangent plane*:

$$y - y_a = -\frac{x_{1a}}{y_a}(x_1 - x_{1a}) - \frac{x_{2a}}{y_a}(x_2 - x_{2a}). \tag{1.29}$$

At the point P_1 : $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, we can form the unique local linearization

$$y - \frac{1}{\sqrt{3}} = -\left(x_1 - \frac{1}{\sqrt{3}}\right) - \left(x_2 - \frac{1}{\sqrt{3}}\right), \tag{1.30}$$

which describes a tangent plane to $f = 0$ at P_1. At the point P_2 : $(-1, 0, 0)$, the equation

$$x_1 = -1 \tag{1.31}$$

describes the tangent plane. But a unique approximation $y(x_1, x_2)$ for the neighborhood of P_2 does not exist on $f = 0$ at P_2 because $y = 0$ there. Of course, a nonunique approximation does exist, that being $y = \pm\sqrt{1 - x_1^2 - x_2^2}$. The sphere defined by $f(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 - 1 = 0$ along with linearizations near P_1 and P_2 is plotted in Figure 1.3.

1.1.3 Many Dependent Variables

Let us now consider the equations

$$f(x, y, u, v) = 0, \tag{1.32}$$

$$g(x, y, u, v) = 0. \tag{1.33}$$

Under certain circumstances, we can unravel Eqs. (1.32, 1.33), either algebraically or numerically, to form $u = u(x, y)$, $v = v(x, y)$. Those circumstances are given by the multivariable extension of the implicit function theorem. We present it here for two functions of two variables; it can be extended to N functions of N variables.

Theorem: Let $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ be satisfied by (x_0, y_0, u_0, v_0), and suppose that continuous derivatives of f and g exist in the neighborhood of

(x_0, y_0, u_0, v_0) with

$$\left.\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}\right|_{x_0,y_0,u_0,v_0} \neq 0. \tag{1.34}$$

Then Eq. (1.34) implies the local existence of $u(x, y)$ and $v(x, y)$.

We will not formally prove this but the following analysis explains its origins. Here we might imagine the two dependent variables u and v both to be functions of the two independent variables x and y. The conditions for the existence of such a functional dependency can be found by local linearized analysis, namely, through differentiation of the original equations. For example, differentiating Eq. (1.32) gives

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv = 0. \tag{1.35}$$

Holding y constant and dividing by dx, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0. \tag{1.36}$$

Operating on Eq. (1.33) in the same manner, we get

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = 0. \tag{1.37}$$

Similarly, holding x constant and dividing by dy, we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0, \tag{1.38}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} = 0. \tag{1.39}$$

The linear Eqs. (1.36, 1.37) can be solved for $\partial u/\partial x$ and $\partial v/\partial x$, and the linear Eqs. (1.38, 1.39) can be solved for $\partial u/\partial y$ and $\partial v/\partial y$ using the well-known Cramer's[4] rule; see Eq. (7.95) or Section A.2. To solve for $\partial u/\partial x$ and $\partial v/\partial x$, we first write Eqs. (1.36, 1.37) in matrix form:

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix}. \tag{1.40}$$

Thus, from Cramer's rule, we have[5]

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} \equiv -\frac{\frac{\partial(f,g)}{\partial(x,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}, \qquad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} \equiv -\frac{\frac{\partial(f,g)}{\partial(u,x)}}{\frac{\partial(f,g)}{\partial(u,v)}}. \tag{1.41}$$

[4] Gabriel Cramer, 1704–1752, Swiss-born mathematician.

[5] Here we are defining notation such as $\partial(f, v)/\partial(u, v)$ in terms of the determinant of quantities, which is easily inferred from the context shown. The notation "≡" stands for "is defined as."

In a similar fashion, we can form expressions for $\partial u/\partial y$ and $\partial v/\partial y$:

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial y} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial y} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} \equiv -\frac{\frac{\partial(f,g)}{\partial(y,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}, \qquad \frac{\partial v}{\partial y} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} \equiv -\frac{\frac{\partial(f,g)}{\partial(u,y)}}{\frac{\partial(f,g)}{\partial(u,v)}}. \tag{1.42}$$

So if these derivatives locally exist, we can form $u(x, y)$ and $v(x, y)$ locally as well. Alternatively, the same procedure has value in simply finding the derivatives themselves.

We take the *Jacobian*[6] *matrix* $\mathbf{J}$ to be defined for this problem as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}. \tag{1.43}$$

It is distinguished from the *Jacobian determinant*, J, defined for this problem as

$$J = \det \mathbf{J} = \frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}. \tag{1.44}$$

If $J \neq 0$, the derivatives exist, and we indeed can find $u(x, y)$ and $v(x, y)$, at least in a locally linear approximation. This is the condition for existence of implicit to explicit function conversion. If $J = 0$, the analysis is not straightforward; typically one must examine problems on a case-by-case basis to make definitive statements.

Let us now consider a simple globally linear example to see how partial derivatives may be calculated for implicitly defined functions.

EXAMPLE 1.4

If

$$x + y + u + v = 0, \tag{1.45}$$

$$2x - y + u - 3v = 0, \tag{1.46}$$

find $\partial u/\partial x$.

We have four unknowns in two equations. Here the problem is sufficiently simple that we can solve for $u(x, y)$ and $v(x, y)$ and then determine all partial derivatives, such as the one desired. Direct solution of the linear equations reveals that

$$u(x, y) = -\frac{5}{4}x - \frac{1}{2}y, \tag{1.47}$$

$$v(x, y) = \frac{1}{4}x - \frac{1}{2}y. \tag{1.48}$$

The surface $u(x, y)$ is a plane and is plotted in Figure 1.4. We could generate a similar plot for $v(x, y)$. We can calculate $\partial u/\partial x$ by direct differentiation of Eq. (1.47):

$$\frac{\partial u}{\partial x} = -\frac{5}{4}. \tag{1.49}$$

[6] Carl Gustav Jacob Jacobi, 1804–1851, German/Prussian mathematician.

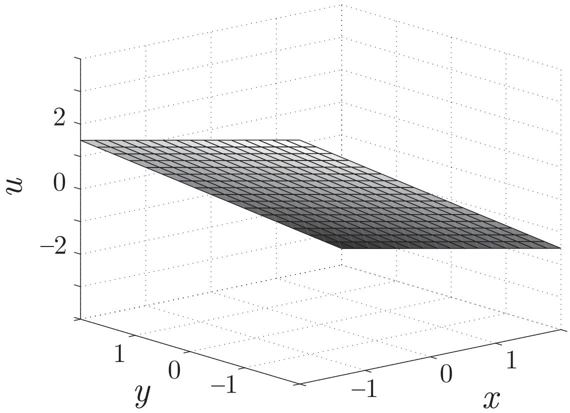

Figure 1.4. Planar surface, $u(x, y) = -5x/4 - y/2$, in the (x, y, u) volume formed by intersection of $x + y + u + v = 0$ and $2x - y + u - 3v = 0$.

Such an approach is generally easiest if $u(x, y)$ is explicitly available. Let us imagine that it is unavailable and show how we can obtain the same result with our more general approach. Equations (1.45) and (1.46) are rewritten as

$$f(x, y, u, v) = x + y + u + v = 0, \tag{1.50}$$

$$g(x, y, u, v) = 2x - y + u - 3v = 0. \tag{1.51}$$

Using the formula from Eq. (1.41) to solve for the desired derivative, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}}. \tag{1.52}$$

Substituting, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -1 & 1 \\ -2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{3 - (-2)}{-3 - 1} = -\frac{5}{4}, \tag{1.53}$$

as expected. Here $J = -4$ for all real values of x and y (also written as $\forall\, x, y \in \mathbb{R}$), so we can always form $u(x, y)$ and $v(x, y)$.

Let us next consider a similar example, except that it is globally nonlinear.

EXAMPLE 1.5

If

$$x + y + u^6 + u + v = 0, \tag{1.54}$$

$$xy + uv = 1, \tag{1.55}$$

find $\partial u/\partial x$.

We again have four unknowns in two equations. In principle, we could solve for $u(x, y)$ and $v(x, y)$ and then determine all partial derivatives, such as the one desired. In practice, this is not always possible; for example, there is no general solution to sixth-order polynomial equations such as we have here (see Section A.1). The two hypersurfaces defined

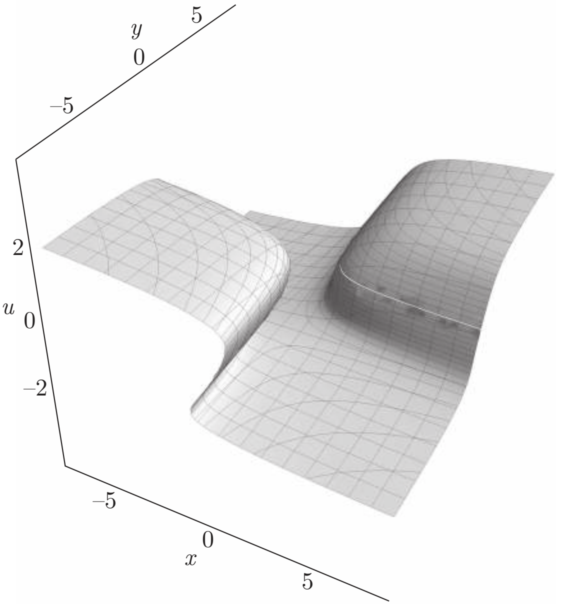

Figure 1.5. Nonplanar surface in the (x, y, u) volume formed by intersection of $x + y + u^6 + u + v = 0$ and $xy + uv = 1$.

by Eqs. (1.54) and (1.55) do in fact intersect to form a surface, $u(x, y)$. The surface can be obtained numerically and is plotted in Figure 1.5. For a given x and y, one can see that u may be nonunique.

To obtain the required partial derivative, the general method of this section is the most convenient. Equations (1.54) and (1.55) are rewritten as

$$f(x, y, u, v) = x + y + u^6 + u + v = 0, \tag{1.56}$$

$$g(x, y, u, v) = xy + uv - 1 = 0. \tag{1.57}$$

Using the formula from Eq. (1.41) to solve for the desired derivative, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}}. \tag{1.58}$$

Substituting, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -1 & 1 \\ -y & u \end{vmatrix}}{\begin{vmatrix} 6u^5 + 1 & 1 \\ v & u \end{vmatrix}} = \frac{y - u}{u(6u^5 + 1) - v}. \tag{1.59}$$

Although Eq. (1.59) formally solves the problem at hand, we can perform some further analysis to study some of the features of the surface $u(x, y)$. Note if

$$v = 6u^6 + u \tag{1.60}$$

that the relevant Jacobian determinant is zero. At such points, determination of either $\partial u/\partial x$ or $\partial u/\partial y$ may be difficult or impossible; thus, for such points, it may not be possible to form $u(x, y)$. Following lengthy algebra, one can eliminate y and v from Eq. (1.59) to arrive at

$$\frac{\partial u}{\partial x} = \frac{1 + 2u^2 + u^7}{-1 + u^2 + 6u^7 - 2ux - 7u^6x - x^2}. \tag{1.61}$$