Cambridge University Press
978-1-107-03540-9 - Stability Regions of Nonlinear Dynamical Systems: Theory, Estimation, and Applications
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Excerpt
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1 Introduction

Stability has been regarded by many as a fascinating and difficult problem of human culture. Stability problems are present in all our lives. The act of standing is an example of maintaining stability that we learn to control at a very early age. Stability of physical systems and of social systems, such as economies and ecosystems, are frequently discussed on the news. Stability is present everywhere and is a fundamental subject that permeates engineering and the sciences.

Stability is a very broad subject, and the concept of stability can be formulated in a variety of ways depending on the intended use of stability analysis and design. As such, at least 50 different terms for stability concepts appear in the literature. An important subject closely related to stability is the *stability region* (i.e. *region of attraction* or *domain of attraction*) of nonlinear dynamical systems, the main subject of this book.

Many nonlinear physical and engineering systems are designed to be operated at an equilibrium state (i.e. equilibrium point). A first and foremost requirement for successful operation of these systems is to maintain stability of this equilibrium state. Stability requires robustness of the equilibrium point to small perturbations, i.e. the system state returns to the equilibrium state after small perturbations. Since most physical and engineering systems are not globally stable, equilibrium states can only be restored under a limited amount (or size) of perturbation. Intuitively, one can state that a system sustaining a larger size of perturbation is “more” stable or “more” robust than another system. We will see that this “degree” of stability is related to the concept of the stability region of nonlinear dynamical systems, which will be thoroughly explained and explored in this book.

1.1 Degree of stability

In order to illustrate the concept of stability and the “degree” of stability or “robustness” of stability, we consider two solid bodies of different size lying on the ground as shown in Figure 1.1. Their respective positions are stable. The stable position of the solid A is recovered after a small perturbation is applied to the solid, as indicated in Figure 1.2. However, if the perturbation is large enough, then the body will reach another equilibrium position, as seen in Figure 1.3, and will not return to the original (stable) equilibrium point. This system possesses multiple equilibrium states, some of which are stable but not globally stable. If one needs to determine which solid has a more

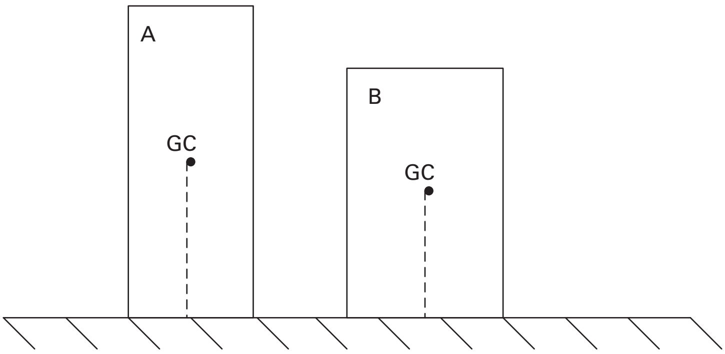

Figure 1.1 Solids A and B have the same weight and volume. The point GC indicates the center of gravity of the solid. Their positions are both stable but the stability of the position of solid B is more robust to perturbations than that of solid A. The stability region of B is larger than the stability region of A.

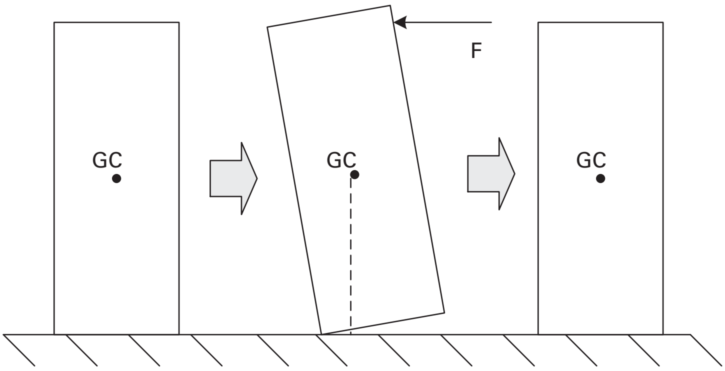

Figure 1.2 The solid is in a stable position. A small perturbation F is applied to the solid. After the removal of that perturbation, the solid returns to its original stable position.

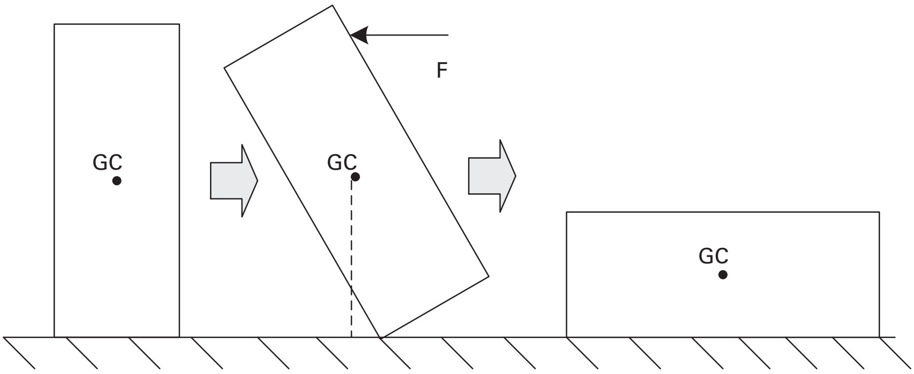

Figure 1.3 The solid is in a stable position. A sufficiently large perturbation F is applied to the solid to make the solid settle down into another stable position, showing that the stable positions of this system are not globally stable.

"stable" equilibrium position, then it is obvious that solid B is "more stable" than solid A. More precisely, the "amount" of disturbance (i.e. perturbation) needed to push solid B away from its stable position is much larger than the amount of disturbance that is needed to push solid A away from its stable position. This "degree of stability" is related to the concept of stability region. It is clear that the stable position of solid B has a larger stability region than the stable position of solid A.

Generally speaking, the concepts of both stability and asymptotic stability are local and do not provide information regarding how robust the system is with respect to disturbance and/or model uncertainty. The concept of stability region, on the other hand, is a global one and gives a complete picture of "the degree of stability" with respect to noise or perturbations or model uncertainty. More specifically, knowledge of the stability region provides, for a given initial condition or a specified amount of perturbation, information on whether or not the system will settle down to a desirable steady state condition.

1.2 Stability regions

We consider the following (autonomous) nonlinear dynamical system

$$\dot{x} = f(x), \quad x \in R^n. \tag{1.1}$$

It is natural to assume the function (i.e. the vector field) $f: R^n \to R^n$ satisfies a sufficient condition for the existence and uniqueness of the solution. The solution of (1.1) starting at x_0 at time $t = 0$ will be denoted $\phi(t, x_0)$, or $x(t)$ when it is clear from the context.

For an asymptotic stable equilibrium point $\hat{x}$, there exists a number $\delta < 0$ such that $\|x_0 - \hat{x}\| < \delta$ implies $\phi(t, x_0) \to \hat{x}$ as $t \to \infty$. In other words, there exists a neighborhood of the equilibrium point such that every solution starting in this neighborhood is attracted to the equilibrium point as time tends to infinity. If δ can be chosen arbitrarily large, then every trajectory is attracted to $\hat{x}$ and $\hat{x}$ is called a *global asymptotically stable equilibrium point*. There are many physical systems containing asymptotically stable equilibrium points but not globally stable equilibrium points. A useful concept for these kinds of systems is that of the **stability region** (also called the **region of attraction** or **domain of attraction**). The stability region of a stable equilibrium point x_s is the set of all points x such that

$$\lim_{t \to \infty} \phi(t, x) = x_s. \tag{1.2}$$

In words, the stability region of x_s is the set of all initial conditions x whose trajectories tend to x_s as time tends to infinity. We will denote the stability region of x_s by $A(x_s)$, and its closure by $\bar{A}(x_s)$, respectively; hence

$$A(x_s) := \{x \in R^n : \lim_{t \to \infty} \phi(t, x) = x_s\}. \tag{1.3}$$

When it is clear from the context, we write A for $A(x_s)$. From a topological point of view, the stability region $A(x_s)$ is an open, invariant and connected set. The boundary of the

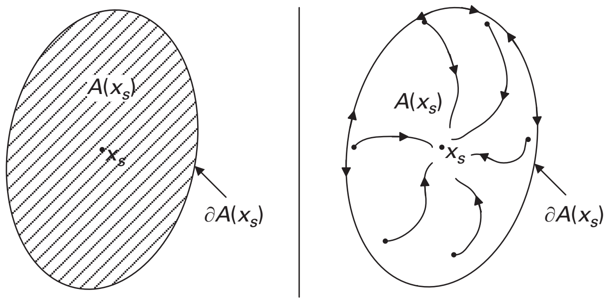

Figure 1.4 As time increases, every trajectory in the stability region $A(x_s)$ converges to the asymptotic stable equilibrium point (SEP) x_s and every trajectory on the stability boundary evolves on the stability boundary.

stability region $A(x_s)$ is called the **stability boundary** (also called the **separatrix**) of x_s and will be denoted by $\partial A(x_s)$. Figure 1.4 illustrates the concept of stability region and stability boundary.

The concept of stability region of an asymptotic stable equilibrium point can be extended to that of other types of attractors. For example, the stability region of an asymptotic stable closed orbit, γ, is defined as follows:

$$A(\gamma) = \{x \in R^n : \lim_{t \to \infty} d(\phi(t, x), \gamma) = 0\}$$

where $d(.,.)$ is a distance function. The stability regions of other types of attractors, such as asymptotically stable quasi-periodic solutions and asymptotically stable chaotic trajectories, are similarly defined. Global stability (i.e. stability in the large) rarely occurs in physical and engineering systems due to physical and operational limits, such as saturations and control actions. In addition, the cost of designing systems to be globally stable, when feasible, is usually very high. These factors make the task of determining stability regions of nonlinear systems of great importance for practical application. It is fair to state that knowledge of the stability region of an attractor is equally important to verification of the stability of the attractor itself.

Knowledge of the stability region is essential in a variety of application areas such as direct methods for power system transient stability analysis [51,65,81,121,207,224,257,264,280], stabilization of nonlinear systems [101,120,147,179,183,227,250,253,279], decentralized control design for nonlinear systems, power system voltage collapse problems [121,126], schemes for choosing manipulator specifications and parameters in robotics [26,196], the design of associative memory in artificial neural networks [71,83,127], solution methods for nonlinear optimization problems [48,49,56,62,162,163], the effect of discretized feedback in a closed loop system on stability [155], and so forth. Knowledge of the stability region of a stable limit cycle is critical in areas such as biology [19] and robotics [274,275].

Determining stability regions of nonlinear dynamical systems is an old problem, and yet it remains challenging. Recent advances in theoretical developments of stability regions and in the development of effective methods for estimating stability regions offer promising results to meet the challenges. These advances can be classified as follows.

Theoretical development:

- characterization of a subset of stability regions of general nonlinear dynamical systems,
- complete characterization of stability boundaries of equilibrium points and attractors of a fairly large class of nonlinear dynamical systems,
- complete characterization of stability boundaries of fixed points of a class of time-discrete nonlinear dynamical systems,
- characterization of stability boundaries of two-time-scale systems,
- complete characterization of stability boundaries of a class of nonlinear non-hyperbolic dynamical systems,
- characterization of relevant stability boundaries of a class of nonlinear dynamical systems.

Methods for estimating the stability region:

- optimal estimation of stability regions of a fairly large class of continuous nonlinear dynamical systems,
- optimal estimation of stability regions of second-order systems,
- optimal estimation of stability regions of a large class of discrete dynamical systems,
- constructive approach to iteratively improve estimations of stability regions,
- optimal estimation of relevant stability regions of a class of nonlinear dynamical systems.

In this book, a comprehensive treatment of these contributions is presented in a structured way, followed by advanced topics on stability regions and by illustrations of stability-region-based practical applications.

1.3 Characterization and estimation of stability regions

It is not easy to trace the first attempt to address the issue of stability regions in the literature, but the history of development of the concept of stability sheds some light on it. The subject of stability has attracted a significant, if not the most, amount of research and development in the area of nonlinear system analysis, design and control. The theory of stability and, in particular, methods to estimate stability regions are related to energy function theory. The mathematicians Leonhard Paul Euler and Joseph-Louis Lagrange, in the eighteenth century, established relations between the equilibrium point and stability with maxima and minima of energy functions. Aleksandr Mikhailovich **Lyapunov** (1857–1918), inspired by his master's thesis on fluid dynamics, developed a general theory of stability in his PhD thesis: *The general problem of the stability of motion* (1892). Lyapunov derived sufficient conditions for stability based on the existence of a scalar function, nowadays called the Lyapunov function in his honor.

George David Birkhoff (1884–1944) made an important contribution by developing stability theory for the asymptotic behavior of trajectories of differential equations.

Krasovskii was probably the first to prove an invariance principle [149]. This principle explores the concept of limit sets introduced by Birkhoff. Joseph P. LaSalle, who received his PhD in 1941, developed a similar invariance principle in 1960 [158]. LaSalle's invariance principle is probably the first tool proposed for estimating stability regions in a systematic way. The work of Krasovskii and LaSalle led to the development of the expression of stability region estimates in the form of level sets of Lyapunov-like functions.This pioneering work has sparked the development of a great number of methods for estimating stability regions of high-dimensional nonlinear systems.

The existing methods for estimating stability regions proposed in the literature can be classified into Lyapunov-function-based methods and non-Lyapunov-function-based methods. Unfortunately, the vast majority of methods always offer rather conservative estimations of stability regions; in other words, these methods offer estimated stability regions which are only a (small) subset of entire stability regions. This conservative estimation can lead to serious consequences. For example, conservative estimates of the stability region may result in unnecessary interruptions in the operation of power systems and in expensive over-design of control systems. Thus, there is a serious need for the development of effective methods for accurately estimating stability regions of high-dimensional nonlinear dynamical systems.

Lyapunov-function-based methods have been popular for estimating the stability regions of stable equilibrium points. This class of methods is applicable to large-scale nonlinear systems but may suffer from overly conservative estimation of stability regions. The degree of conservativeness of Lyapunov-function-based methods in estimating stability regions depends on the underlying Lyapunov function and the associated value of the critical level. Finding a good Lyapunov function for estimating stability regions is not an easy task. There is no systematic way to derive a Lyapunov function for general nonlinear dynamical systems. Recent advances along this line of research and development include the proposal of maximal Lyapunov functions [256], the optimal estimation of stability regions based on a given Lyapunov function [53], LMI optimization techniques for constructing Lyapunov functions, and estimating stability regions for polynomial dynamical systems [41,76,137,164,247,250,252] and for non-polynomial systems [39]. These optimization-based methods entail high computation effort, tending to grow rapidly with the system dimension, making them unsuitable for estimating stability regions of high-dimensional nonlinear dynamical systems. The LMI-based estimation methods generally yield (overly) conservative estimations of stability regions.

Another advance is the constructive Lyapunov function methodology for determining optimal Lyapunov functions to estimate stability regions [60]. The constructive methodology yields a sequence of estimated stability regions which form a strictly monotonic increasing sequence, and yet each of them is contained in the stability region of the system under study. The constructive methodology can either stand by itself or be used with existing methods serving as the input to the constructive methodology. This methodology can reduce the conservativeness in estimating stability regions via the Lyapunov function approach.

There are very few methods which are able to estimate the entire stability region. They all entail serious computational problems, making their application impractical. For example, Zubov's method [106,175] offers a technique for computing the entire stability region via the "optimal" Lyapunov function. However, constructing this "optimal" Lyapunov function requires solving a set of nonlinear partial differential equations (PDEs) which are difficult, if not impossible, to solve. Because of this problem, several techniques have been proposed which attempt to approximate the solution of the PDEs, but with limited success. Furthermore, it has been found that even for some second-order systems the use of approximated solutions to construct the "optimal" Lyapunov function still results in rather conservative estimations.

Another method which attempts to reduce the conservativeness in estimating the stability boundary was proposed in [97], in which the estimated stability boundary was synthesized from a number of system trajectories obtained by backward integration. This method, known as the trajectory reversing method [97,99,168], is suitable only for low-dimensional systems due to its excessive computational burden. Other methods for estimating the stability region include approximating the stability boundary by polytopes [208] and the cell-to-cell mapping method [128].

A majority of the existing proposed methods for estimating stability regions are only applicable to low-dimensional nonlinear systems. Very few of them are applicable to high-dimensional nonlinear systems (say several hundred or thousands of dimensions). To be practical, estimation methods need to be able to deal with very large-scale systems (say, tens of thousands of state variables).

An important breakthrough in the development of the theory of stability regions occurred in the 1980s with the formulation of a comprehensive theory of stability regions and a complete characterization of the stability regions of a class of nonlinear dynamical systems [54]. This class of nonlinear dynamical systems is characterized by its ω-limit set being composed of equilibrium points and limit cycles. A conceptual method based on the complete characterization derived in [54], when feasible, can find the *entire* stability region. This method is based on a complete characterization of the stability boundary via the stable manifolds of unstable equilibrium points and unstable limit cycles on the stability boundary. For low-dimensional nonlinear dynamical systems, the derivation of the stable manifolds can be achieved by numerical methods. However, current computational methods are inadequate for computing the stable manifolds of unstable equilibrium points and/or of unstable limit cycles of high-dimensional nonlinear dynamical systems.

Knowledge of the complete characterization of a stability boundary has led to the development of several effective methods for accurate estimation of stability regions. By exploring the complete characterization of the stability boundary, several computational schemes for optimally estimating the stability regions have been developed, see for example [53]. These complete characterizations have also led to the development of effective methods to estimate relevant parts of the stability boundary. These methods have been fundamental to advancing a variety of important practical applications.

1.4 Practical applications of stability regions

Estimating or determining stability regions is central to solving many problems arising in sciences and engineering. The following list is a non-exhaustive enumeration of applications where knowledge of stability regions plays an important role.

1. Biology:
   - micromolecules and macromolecules;
   - dynamics of ecosystems.
2. Biomedicine:
   - dynamics of the immune response;
   - human respiratory models.
3. Control:
   - sliding control systems;
   - control of nonlinear systems;
   - linear systems with saturated controls;
   - control of polynomial systems.
4. Economics:
   - economic growth rate;
   - carrying capacity of the human population.
5. Robotics:
   - asymptotically stable walking cycle of a bipedal robot;
   - regulator design of robot manipulators.
6. Power grids and power systems:
   - direct power system transient stability analysis;
   - dynamic voltage stability analysis.
7. Power electronic circuits:
   - power-electronic-based converters;
   - DC-to-DC converters.
8. Neural networks:
   - dynamic recurrent neural networks;
   - associative memory;
   - Hopfield neural network models.
9. Optimization problems:
   - unconstrained optimization problems;
   - constrained optimization problems.

We briefly describe some of the above applications based on knowledge of stability regions.

1.4.1 Immune response

Stability regions provide insight into the analysis of the immune response where interaction among lymphocyte populations, antigens and antibiotics occurs [110,139,225,267]. The coexistence of multiple equilibriums has been reported in these models and the outcome of a treatment depends not only on how antigens are administered but also on the initial condition of the system when the treatment began. Depending on the initial condition, the dynamics of the treatment will converge to the attractor whose stability region contains the initial condition.

The immune system is primarily composed of a large number of cells called lymphocytes. Lymphocytes produce antibodies that bind to invading organisms (antigens) in order to eliminate them. An animal can produce a very large number of different antibodies (10^6 to 10^7) [110]. The presence of an antigen stimulates the proliferation of cells (lymphocytes) with the specific antibody for that invader. The following set of differential equations models the dynamics of positive and negative cells of the immune system:

$$\dot{x}_+ = x_+[R(x_-, x_+) - D(x_-, x_+) - k_4] + k_5$$
$$\dot{x}_- = x_-[R(x_+, x_-) - D(x_+, x_-) - k_4] + k_5$$

where x_+ and x_- respectively represent the concentrations of positive and negative cells in the organism. The constant k_5 is an influx rate term while k_4 is a death rate term. The function R is a replication rate that describes the proliferation of the positive cells, while the function D is a death rate due to killing by antibodies or anti-antibodies. A typical phase portrait of the immune dynamical system is shown in Figure 1.5. Typically, immunization systems have four asymptotically stable equilibrium points: the virgin state, indicated as

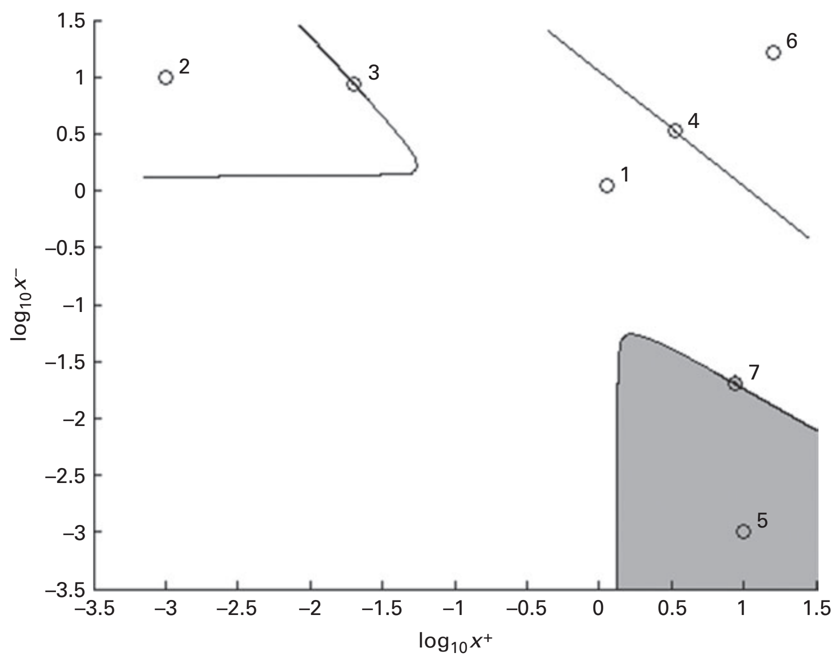

Figure 1.5 The stability region of the immune state (equilibrium #5) of a typical immunization system is highlighted. The immune system contains four asymptotically stable equilibrium points (SEPs): equilibriums #1 (virgin state), #2 (anti-immune state), #5 (immune state) and #6 (suppressed state). Their stability boundaries are depicted in this figure as black lines. Equilibriums #3, #4 and #7 are unstable equilibrium points lying on the stability boundaries of these four SEPs.

equilibrium #1 in Figure 1.5, where the concentrations of both positive and negative cells are low, the immune state, depicted as equilibrium #5, where the concentration of positive cells is much larger than that of negative cells, the anti-immune state, equilibrium #2, where the concentration of negative cells is high compared to the concentration of positive cells, and a suppressed state, equilibrium #6, where both concentrations are high. The immunization problem consists of designing a proper perturbation to the immune system, an immunization shot for example, such that the initial condition of the system after perturbation lies inside the stability region of the immune state equilibrium point, highlighted in gray in Figure 1.5. As a result, the immune system of the organism will abandon the virgin state equilibrium #1 and settle down in the immune state equilibrium #5.

1.4.2 Ecosystems

In biology, knowledge of stability regions is relevant to the problems of ecosystem dynamics in which different species coexist, in particular the coexistence of predators and prey species [19,148,176]. For example, the influence of commercial exploitation of a population of salmon on the size of stability regions was investigated in [206]. The resilience of an ecosystem can be viewed as the problem of determining whether or not a certain initial population lies inside the stability region of an attractor. Large stability regions are usually related to high-level "resilience" of the ecosystems.

1.4.3 Micromolecules and macromolecules

The task of finding saddle-points lying on a potential energy surface plays a crucial role in understanding the dynamics of micromolecules as well as in studying the folding pathways of macromolecules such as proteins. This task has been a topic of active research in the field of computational chemistry for more than two decades. Several methods have been proposed in the literature based on the Hessian matrix at the saddle-points, see for example [21,141]. These methods are not applicable to high-dimensional problems because the computational cost increases tremendously as the system dimension increases. Due to the scalability issue, several first-derivative-based methods for computing the saddle-points have been proposed, see for example [189,210]. A detailed description of such methods, along with their advantages and disadvantages, can be found in a survey paper [119]. A stability-region-based method has been developed for finding saddle-points of high-dimensional problems [213,214]. This method is based on the transformation of the task of finding the saddle-points into the task of finding the dynamic decomposition points lying on the stability boundary of two local minima (i.e. two stable equilibrium points). This method does not require that the gradient information starts from a local minimum (i.e. a stable equilibrium point). It finds the stability boundary in a given direction and then traces the stability boundary until the dynamic decomposition point (i.e. the saddle-point) is reached. This tracing of the stability boundary is far more efficient than searching for saddle-points in the entire search space. This again illustrates that knowledge of the stability boundary plays a key role in the development of this effective computational method.