Introduction

In the study of higher categories, dimension three occupies an interesting position on the landscape of higher dimensional category theory. From the perspective of a "hands-on" approach to defining weak n-categories, tricategories represent the most complicated kind of higher category that the community at large seems comfortable working with. On the other hand, dimension three is the lowest dimension in which strict n-categories are genuinely more restrictive than fully weak ones, so tricategories should be a sort of jumping off point for understanding general higher dimensional phenomena. This work is intended to provide an accessible introduction to coherence problems in three-dimensional category.

1 Tricategories

Tricategories were first studied by Gordon, Power, and Street in their 1995 AMS Memoir. They were aware that strict 3-groupoids do not model homotopy 3-types, and thus the aim of their work was to create an explicit definition of a weak 3-category which would not be equivalent (in the appropriate three-dimensional sense) to that of a strict 3-category. The main theorem of Gordon *et al.* (1995) is often stated: every tricategory is triequivalent to a **Gray**-category. Triequivalence is a straightforward generalization of the usual notion that two categories are equivalent when there is a functor between them which is essentially surjective, full, and faithful. The new and interesting feature of this result is the appearance of **Gray**-categories. These are categories which are enriched over the monoidal category **Gray**; this monoidal category has the category of 2-categories and strict 2-functors as its underlying category, but its monoidal structure is not the Cartesian one. **Gray**-categories can thus be viewed as a maximally strict yet still completely general form of weak 3-category, and it is known, for instance, that **Gray**-groupoids model all homotopy 3-types.

My interest in tricategories began while carrying out joint work with Eugenia Cheng on the Stabilization Hypothesis of Baez and Dolan. The Stabilization Hypothesis roughly states that k-degenerate weak $(n + k)$-categories correspond to what they called k-tuply monoidal n-categories. Here, k-degenerate means that the $(n + k)$-categories only have a single 0-cell, single 1-cell, and so on, up to having only a single $(k - 1)$-cell: thus the bottom k dimensions are degenerate. A k-tuply monoidal n-category is one which is monoidal, and as k increases that monoidal structure becomes more and more commutative until it stabilizes when $k = n + 2$. Some relevant examples to keep in mind are

- the case $k = 1, n = 0$ gives 1-degenerate categories (categories with a single object) on the one hand or 1-tuply monoidal 0-categories (sets with an associative and unital multiplication) on the other hand; and
- fixing $n = 1$ we get weak 2-categories with a single object, weak 3-categories with a single object and single 1-cell, and weak 4-categories with a single object, 1-cell, and 2-cell on the one hand and monoidal categories, braided monoidal categories, and symmetric monoidal categories on the other hand.

The Stabilization Hypothesis is a guiding principle of higher category theory, yet we found that no systematic study of low dimensional cases had been carried out.

As had already been discovered by Tom Leinster, k-degenerate $(n + k)$-categories and k-tuply monoidal n-categories were not precisely the same structures, at least when using the explicit, algebraic notions of weak n-category. As an example, a bicategory with a single object and single 1-cell is not only a commutative monoid given by the set of 2-cells $I \Rightarrow I$ under composition (where I is the single 1-cell), but is in fact a commutative monoid equipped with a distinguished invertible element. This element corresponds to the left (or right, they are equal) unit isomorphism, and satisfies no axioms. So in fact it is the *algebraic nature* of the definition of bicategory that creates this extra piece of data. To carry out the same analysis in dimension three, we needed a fully algebraic definition of tricategory, and the definition of Gordon, Power, and Street was only partially algebraic.

The original definition was partially algebraic because it included data having certain properties but not the data necessary to check those properties. In particular, the associativity equivalence for 1-cell composition is a 2-cell

$$a_{h,g,f} : (h \otimes g) \otimes f \Rightarrow h \otimes (g \otimes f),$$

but the original definition did not include a 2-cell $a^{\bullet}_{h,g,f}$ nor invertible 3-cells $1 \cong a^{\bullet}_{h,g,f} \circ a_{h,g,f}, a_{h,g,f} \circ a^{\bullet}_{h,g,f} \cong 1$ verifying that the 2-cell $a_{h,g,f}$ was an equivalence. While this seems like a minor technical point, it does have an impact on how one goes about manipulating tricategories and the cells between them. Making an algebraic definition was necessary for an examination of the structures in the Stabilization Hypothesis, but one also requires a choice of the cells $a^{\bullet}_{h,g,f}$ in order to define a composition law on transformations between functors of tricategories.

These concerns led me to consider a fully algebraic definition of tricategory in my 2006 University of Chicago Ph.D. thesis. While the changes to the definition are minor, they do allow the definition of more constructions on tricategories such as functor tricategories and an explicit strictification. The most important difference from the partially algebraic case is how coherence is approached. While both proofs of coherence for tricategories involve embedding a tricategory in a **Gray**-category, the fully algebraic definition makes more direct use of a Yoneda embedding, much like how coherence for bicategories is usually proved. Continuing to employ techniques similar to those used in the case of bicategories, it is also possible to use the fully algebraic definition to prove a coherence theorem for functors.

Tricategories have appeared in more applications recently, particularly in topological applications. Carrasco, Cegarra, and Garzón (2011) study a Grothendieck construction for diagrams of bicategories (of which tricategories are an example) in order to understand the classifying spaces of braided monoidal categories. Lack (2011) has constructed a model category structure on the category of **Gray**-categories and **Gray**-functors that restricts to a model structure on **Gray**-groupoids. With these model structures in hand, Lack goes on to prove that **Gray**-groupoids model homotopy 3-types. My paper (Gurski 2011) proves a coherence theorem for braided monoidal bicategories that uses tricategorical techniques in a number of ways.

2 Gray-monads

The study of **Gray**-monads and their algebras has two distinct sides, reminiscent of the study of 2-monads. First, **Gray**-monads are just monads enriched over the monoidal category **Gray**, and thus carry with them the usual structure associated to enriched monads. The category **Gray** of 2-categories and 2-functors, but equipped with the **Gray**-tensor product, has many pleasant properties so we can reproduce many of the usual constructions from monad theory such as Eilenberg–Moore objects for a **Gray**-monad. The second half of the story for **Gray**-monads is the three-dimensional picture, consisting of many different kinds of algebras and maps that all take advantage of the higher dimensional nature of a **Gray**-category. This side of the picture is much more complicated in terms of data and axioms, but the objects that arise from it are much more interesting from the perspective of applications in other parts of higher dimensional category theory. Comparing these two aspects of the theory of **Gray**-monads is the study of a very general kind of coherence question.

This form of coherence goes back to the seminal work *Two-dimensional Monad Theory* by Blackwell, Kelly, and Power (1989). That paper was concerned with 2-monads, and studied the two-dimensional aspects using the more widely understood **Cat**-enriched theory for comparison. The basic situation was as follows. Let A be a 2-category, and T a 2-monad (i.e., **Cat**-enriched monad) on it; a simple example to keep in mind is when $A = \mathbf{Cat}$ and T is the 2-monad for strict monoidal categories. We can now form (at least) three different 2-categories: the 2-category A^T which is the Eilenberg–Moore object in the enriched sense, the 2-category T-Alg of algebras with pseudo-algebra morphisms, and the 2-category T-Alg$_l$ of algebras with lax algebra morphisms. Each of these 2-categories has the same objects, and there are inclusions

$$A^T \hookrightarrow T\text{-Alg} \hookrightarrow T\text{-Alg}_l$$

which are locally full on 2-cells. The first main result of Blackwell *et al.* (1989) is that, under some conditions on A and T, the inclusions

$$A^T \hookrightarrow T\text{-Alg}, \quad A^T \hookrightarrow T\text{-Alg}_l$$

each have a left 2-adjoint. The image of an object X under this left adjoint is often denoted X', and the one-dimensional aspect of this 2-adjunction states that "weak" algebra maps (either pseudo-algebra morphisms or lax algebra morphisms, depending on the particular example) $X \to Y$ are in bijection with algebra morphisms $X' \to Y$ in the usual sense of monad theory.

What I have described so far is in fact the most basic situation, and we can consider more complicated scenarios in which not only are the morphisms allowed to be weakened, but so is the notion of algebra as well. Once again there will be an inclusion of A^T into whatever 2-category of algebras we choose to study, and it is possible to give conditions under which this inclusion has a left 2-adjoint $X \mapsto X'$. The unit of this adjunction will be a morphism $X \to X'$, and it is also possible to give conditions under which these components are equivalences. In other words, this very abstract form of coherence can often be used to derive the usual kinds of coherence theorems such as coherence for monoidal categories.

The conditions on the 2-category A and the 2-monad T to ensure that these inclusions have a left adjoint, and then perhaps to show that the unit of the adjunction has components which are equivalences, are conditions about the existence of certain kinds of two-dimensional limits and colimits in A together with the requirement that T preserve some of these. The most complete treatment of this perspective can be found in *Codescent Objects and Coherence* by Steve Lack (2002a). In this paper, Lack shows how the most important colimit to consider is that of the codescent object which plays the role of a kind of two-dimensional coequalizer. Understanding codescent object turns out to be essential in studying coherence through this kind of strategy (i.e., by constructing a left adjoint to the inclusion of the "strict algebra case" into some larger 2-category with weaker objects and/or morphisms), and leads to theorems about the existence of the left adjoint as well as showing the components of the unit are equivalences.

Far less has been studied in the three-dimensional world. The only work thusfar in this direction is a paper of John Power's (2007) in which he begins the study of **Gray**-monads and their algebras. Here, the basic objects of study are **Gray**-categories equipped with a **Gray**-monad; examples are much harder to come by, but one to keep in mind is that of 2-categories equipped with a choice of flexible limits or colimits. The work of Power should be seen as the analogue of many parts of the original paper of Blackwell–Kelly–Power, and he proves many of the same basic theorems. He establishes the notions of weak or lax algebra maps, together with the higher cells between them, and proves that these form a **Gray**-category containing the usual enriched category of algebras. Under some cocompleteness conditions, he proves that the inclusion of algebras with strict maps into algebras with weak maps has a left adjoint, and using pseudo-limits of arrows he gives a sufficient condition for the unit of this adjunction to have components which are internal biequivalences. He does not, however, pursue these using codescent techniques, but does remark that such a strategy might be useful for a complete understanding of coherence problems in dimension three.

3 An outline

This book is aimed at being a basic guide to coherence problems in three-dimensional category theory. From the above discussion, it should not be surprising that this is split into two parts. In the first part, we will discuss the coherence theorem for tricategories and the related result for functors; much of this material has been adapted from my 2006 Ph.D. thesis. The second part focuses on the general coherence problem for algebras over a **Gray**-monad using codescent methods. Just as Lack's paper can be seen as a refinement of the basic results in Blackwell–Kelly–Power, the results in the second half of this work can be seen as a refinement of Power's (2007) results. It is the intention that this book can be read without any prior experience with tricategories or **Gray**-categories, and I have included background material in an attempt to keep this book self-contained. The only exception is the inclusion of some calculational results that were proved by Gordon *et al.* (1995) and are of general use in the proofs leading up to the coherence theorem for tricategories. Most of these calculations are omitted because of the size of the diagrams involved so it might not be clear how these results might be used, but they are quite useful for performing many of these computations. Here is a detailed outline of what is to come.

First, I will give some background information and establish notation. Since tricategories and **Gray**-categories have three different composition operations on 3-cells, it is important to establish clear notation early on. With this in mind, I will use some slightly non-standard notation even at the level of bicategories which can then easily be augmented when moving to the three-dimensional world later on. It is also important to keep in mind that at each dimension there are choices to be made about the canonical direction of the data present in many different definitions. With this in mind, I will follow Gordon–Power–Street in using the oplax direction for transformations as the default notion although in practice this has little bearing since we will be more interested in the pseudo-natural rather than the lax case. I will also recall the concept of an icon, and remind the reader of the necessary calculational results from the theory of mates that will be useful later.

The second piece of background material I will discuss is coherence for bicategories. I will present a number of formulations of this theorem, and will follow the strategy used by Joyal and Street (1993) to prove these different incarnations of coherence. Their approach provides a solid framework for proving coherence for functors as well, and it is this feature in particular that will be important later as the original work of Gordon–Power–Street did not have a proof of coherence for functors between tricategories.

The final section of background will be a discussion of the **Gray**-tensor product and **Gray**-categories. I will present many different ways of thinking about the **Gray**-tensor product, but will give very few proofs. My goal is less to give a fully rigorous account of this monoidal structure on the category of 2-categories and 2-functors and more to provide the reader with a basic understanding coupled with some intuition on how to manipulate these structures. **Gray**-categories will feature prominently in the rest of this work, and while the rules for working in a **Gray**-category are not much more complicated than those for working in a strict 2- or strict 3-category, there are some important differences to keep in mind while doing calculations inside an arbitrary **Gray**-category.

With the background completed, we are ready to move on to discussing tricategories and their coherence theory. I will begin with the relevant definitions of tricategories and the higher cells between them. It is at this point that we diverge slightly from the treatment in Gordon–Power–Street, as the definition I will give has a bit more structure than the one they work with. The specific difference between the two definitions is that they require certain transformations to be equivalences, while I specify an entire adjoint equivalence as part of the data. This difference does not change the definition in a conceptual way, but does make more techniques available.

Next I give some basic examples of tricategories and functors between them. The most important examples are **Bicat** and **Gray**, and they occupy the first part of this chapter. These examples will be used later in the proof of coherence, and so are worth constructing in detail. Then I give a topological example which, to my knowledge, has not been explored in the literature thusfar: the fundamental 3-groupoid of a space. This is a straightforward construction, but important in studying the relationship between three-dimensional groupoids and homotopy 3-types.

The next chapter is devoted to a discussion of the many different kinds of free objects that arise in this theory. There are at least four different types of graphs from which we can generate free tricategories or **Gray**-categories, and this section is devoted to cataloguing all of the free constructions on these different types of underlying data. It is actually at this point that the change in the definition of tricategory makes its (technical) appearance, as it is simple to freely generate an adjoint equivalence while it is not clear what it would mean to freely generate an equivalence. This chapter also begins the discussion of the category of tricategories and strict functors; this requires some care, as the composition law in this category does not give the same result as the composition of strict functors *qua* weak functors.

I will then discuss some of the basic constructions that would go into making a weak four-dimensional category **Tricat**. In particular, I will give constructions of some composites of higher cells. Since this is largely a matter of bookkeeping, I will only define the composites that we need later; thus the first obstruction to finishing the definition of **Tricat** is to define a few more kinds of composition. The second obstruction is providing all the rest of the data, including things like associators and unit constraints for each of the different levels of composition. This has to all be packaged to give a composition *functor* between tricategories, with associativity and unit transformations, and so on, and each piece of data here has components which are themselves transformations, etc., at which point it becomes clear that constructing **Tricat** by hand, without any tools, is a huge task that will, most likely, not produce fruit in proportion to the work required (at least at this stage in the development of the theory). I would also like to point out that the changes in the definitions that I have made affect this section as well. Defining some of these composites actually requires using the pseudo-inverses of the data in the Gordon–Power–Street definition of a tricategory, so Gordon *et al.* (1995) defined these composites only up to some ambiguity. This is the benefit of making the definitions fully algebraic: whenever you want to define a new construction, every piece of data you might want is already on hand. The downside, of course, is that the things you are defining become much more complicated. In this case, though, the complications are all of a *computational* rather than *conceptual* nature, and I believe that the drawback of having longer definitions is offset by being able to follow a more satisfying strategy for proving coherence.

The next chapter details how **Gray**-categories can be seen as examples of tricategories. Here I will also explore the intermediate notion of a cubical tricategory. This notion is important because it provides a stepping stone in the proof of coherence. The simplest proof that every bicategory is biequivalent to a 2-category employs the Yoneda embedding together with the fact that every functor bicategory of the form $[X, K]$, where K is a 2-category, is itself a 2-category. Since the Yoneda embedding lands in a functor bicategory $[B^{\mathrm{op}}, \mathbf{Cat}]$, and **Cat** is a 2-category, the strictification result follows, albeit without a particularly explicit construction of how to strictify a given bicategory. If we tried to follow the same idea for tricategories, we would land in a functor tricategory $[T^{\mathrm{op}}, \mathbf{Bicat}]$, but since **Bicat** is not a **Gray**-category, this would not produce the desired result. Thus we seem to need a bit more structure on the tricategory T to use Yoneda for the proof of coherence, and this extra structure is that of a cubical tricategory.

With this Yoneda-style proof in mind, Chapter 9 begins with the construction of functor tricategories when the target is a **Gray**-category. Since the functor tricategory inherits the compositional structure of the target, it also becomes a **Gray**-category. We will see this directly, although Power (2007) also notes that something very close to this structure can be constructed using pseudo-algebras for a particular **Gray**-monad. I will also note that this corrects a mistake of Crans (1999). Finally, it is time to construct an appropriate Yoneda embedding. Here we restrict ourselves to the case that the tricategory in question is cubical, as this will produce a Yoneda embedding of the form $T \hookrightarrow [T^{\mathrm{op}}, \mathbf{Gray}]$, which by previous results is a **Gray**-category itself.

The full coherence result is then obtained by composing the Yoneda embedding with a canonical triequivalence $S \to \operatorname{st} S$ from a generic tricategory S to a cubical one. This triequivalence, and the cubical nature of $\operatorname{st} S$, both arise directly from an explicit strictification that is a consequence of coherence for bicategories. In the search for a good notion of semi-strict 4-category, I would argue that understanding the properties of explicit strictifications of tricategories is the key to extending the strategy here up one dimension.

The next chapter analyzes the question of coherence from the perspective of free constructions. Using the Yoneda-embedding version of coherence, I prove that the free strict 3-category, free **Gray**-category, and free tricategory on a 3-globular set are all triequivalent. From this, we obtain the corollary that any diagram of coherence 3-cells in a tricategory which arises from a diagram of coherence 3-cells in a *free* tricategory must commute. As a non-example, I explain how the "categorified Eckmann–Hilton argument" produces a diagram which is not required to commute, and in general does not: any braided, but not symmetric, monoidal category gives an example of a tricategory in which it fails to commute (see Cheng and Gurski (2011) for a rigorous discussion of the relationship between braided monoidal categories and doubly degenerate tricategories). I then prove a coherence theorem for functors, once again following the strategy of Joyal and Street. This is a result which was not possible using the techniques in Gordon–Power–Street. As a consequence of this theorem, it is possible to unambiguously interpret the diagrams in Trimble's webpage definition of a tetracategory, much as how the definition of a tricategory requires the use of coherence for functors between bicategories to be interpretted in a rigorous fashion. Finally, I consider the problem of finding an explicit strictification functor. This strictification is constructed directly, and I will show that it strictifies functors as well. It has a computadic flavor, but that angle is not pursued any further here. It seems likely that a cleaner approach could be made, but that it might require using an unbiased notion of tricategory from the beginning.

Part III of this work concerns the general coherence problem for lax and pseudo-algebras over a **Gray**-monad. The first chapter in this section provides the basic definitions of lax codescent diagrams, codescent diagrams, and the (lax or pseudo) codescent objects associated to each. Codescent diagrams should be considered higher dimensional versions of coequalizers, and every algebra over a monad is, in a canonical fashion, a coequalizer of free algebras. This fact is a generalization of the simple result that every group is a quotient of a free group, i.e., every group can be given a presentation, and is central in the study of monads and their algebras. The domains of these codescent diagrams are constructed explicitly, and in the lax case are given by the **Gray**-category associated to a **Gray**-computad (Batanin 1998b); the pseudo case can be expressed as the **Gray**-category associated to a **Gray**-computad modulo an equivalence relation on parallel 3-cells exhibiting that certain generators are inverse to each other.

The second chapter is a further exploration of the notion of codescent, this time as a weighted colimit. I have included a short reminder on weighted colimits in the context of **Gray**-categories, as well as some examples which help build to the notion of a codescent object. Just as in the two-dimensional case, it is not very difficult to show that codescent objects can be constructed from more basic weighted colimits. This chapter is largely of theoretical importance, as much is already known about weighted colimits in the general case of $\mathcal{V}$-enriched categories. Thus being able to express codescent objects as the colimits for a certain weight allows them to be studied using the general, and well-developed, machinery of enriched categories. In fact, that statement encapsulates the general philosophy of Part 3: we use enriched category theory as much as possible in order to relate weak algebras for a monad to the strict ones. This was the philosophy championed by Max Kelly, and it is applicable at the three-dimensional level in many of the same ways that it was applicable to the study of algebras over 2-monads.

The next chapter deals with constructing **Gray**-categories of algebras for a given **Gray**-monad. I first follow the standard theory of enriched monads to construct the enriched Eilenberg–Moore object; this is the **Gray**-category whose objects are strict algebras, and whose higher cells are completely strict versions of functors, transformations, and modifications. I also give the definitions of the lax and pseudo versions of these, and construct appropriate **Gray**-categories of each. It should be noted that, in both the lax and pseudo cases, the 2-cells are the pseudo-strength version of transformations in each. This requirement follows from using the pseudo-strength version of the **Gray**-tensor product rather than the lax version in Gray's (1974) original work. I believe all of the definitions can be modified for the situation of the lax tensor, but at the moment I know of no applications that would benefit from that alteration. Many of these definitions can be found, albeit with some slightly different conventions on direction, in the paper of Power (2007).

We have now laid the foundations for work on the general coherence problem, and Chapter 14 attacks this by using codescent objects to construct left adjoints from the **Gray**-categories of lax and pseudo-T-algebras into the **Gray**-category of strict algebras. This proceeds, much as in Lack's work in the two-dimensional case, by first constructing a canonical codescent diagram from an algebra, and then showing that its codescent object gives a left adjoint. I prove this style of theorem in the lax case, as the pseudo-strength version